

# On an inhomogeneous slip-inflow boundary value problem for a steady flow of a viscous compressible fluid in a cylindrical domain

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July 24, 2009

## Abstract

We investigate a steady flow of a viscous compressible fluid with inflow boundary condition on the density and inhomogeneous slip boundary conditions on the velocity in a cylindrical domain  $\Omega = \Omega_0 \times (0, L) \in \mathbb{R}^3$ . We show existence of a solution  $(v, \rho) \in W_p^2(\Omega) \times W_p^1(\Omega)$ , where  $v$  is the velocity of the fluid and  $\rho$  is the density, that is a small perturbation of a constant flow  $(\bar{v} \equiv [1, 0, 0], \bar{\rho} \equiv 1)$ . We also show that this solution is unique in a class of small perturbations of  $(\bar{v}, \bar{\rho})$ . The term  $u \cdot \nabla w$  in the continuity equation makes it impossible to show the existence applying directly a fixed point method. Thus in order to show existence of the solution we construct a sequence  $(v^n, \rho^n)$  that is bounded in  $W_p^2(\Omega) \times W_p^1(\Omega)$  and satisfies the Cauchy condition in a larger space  $L_\infty(0, L; L_2(\Omega_0))$  what enables us to deduce that the weak limit of a subsequence of  $(v^n, \rho^n)$  is in fact a strong solution to our problem.

## 1 Introduction

The mathematical description of a flow of a viscous, compressible fluid usually lead to problems of mixed character as the momentum equation is elliptic (in stationary case) or parabolic (in case of time-dependent flow) in the velocity, while the continuity equation is hyperbolic in the density. Therefore, the application of standard methods usually applied to elliptic or hyperbolic problems fails in the mathematical analysis of the compressible flows and a combination of such techniques, as well as development of new mathematical tools is required. As a result a consistent theory of weak solutions to the Navier - Stokes equations for compressible fluids has been developed quite recently in the 90's, mainly due to the work of Lions [11] and Feireisl [6]. An overview of these results is given in the monograph [16]. A modification of this approach in case of steady flows with slip boundary conditions has been developed by Mucha and Pokorný in a two dimensional case in [14] and in 3D in [23].

The issue of regular solutions is less investigated and the problems are considered mainly with Dirichlet boundary conditions. If we assume that the velocity does not vanish on the boundary, the hyperbolicity of the continuity equation makes it necessary to prescribe the density on the part

of the boundary where the flow enters the domain. In [25] Valli and Zajackowski investigate a time-dependent system with inflow boundary condition, obtaining also a result on existence of a solution to stationary problem. The existence of regular solutions to stationary problems with an inflow condition on the density has been investigated by Kellogg and Kweon [8] and Kweon and Song [10]. Their results require some smallness assumptions on the data, and the regularity of solutions is a subject to some constraints on the geometry of the boundary near the points where the inflow and outflow parts of the boundary meet. In [9] Kellogg and Kweon consider a domain where the inflow and outflow parts of the boundary are separated, obtaining regular solutions.

The lack of general existence results inhibits the development of qualitative analysis of compressible flows. Therefore it is worth to mention here the papers by Plotnikov and Sokolowski who has investigated shape optimization problems with inflow boundary condition in 2D [21] and 3D [22] dealing with weak solutions. More recently Plotnikov, Ruban and Sokolowski have investigated shape optimization problems working with strong solutions in [19] and [20].

It seems interesting both from the mathematical point of view and in the eye of applications to investigate problems with inflow boundary condition on the density combined with slip boundary conditions on the velocity, that enables to describe precisely the action between the fluid and the boundary. Such problem is investigated in this paper. The domain is a three dimensional cylinder and we assume that the fluid slips along the boundary with a given friction coefficient and there is no flow across the wall of the cylinder. We show existence of a regular solution that can be considered a small perturbation of a constant solution. The method of the proof is outlined in the next part of the introduction and now we are in a position to formulate our problem more precisely.

The flow is described by the Navier-Stokes system supplied with the slip boundary conditions on the velocity. The complete system reads

$$\begin{aligned}
\rho v \cdot \nabla v - \mu \Delta v - (\mu + \nu) \nabla \operatorname{div} v + \nabla \pi(\rho) &= 0 && \text{in } \Omega, \\
\operatorname{div}(\rho v) &= 0 && \text{in } \Omega, \\
n \cdot \mathbf{T}(v, \pi(\rho)) \cdot \tau_k + f v \cdot \tau_k &= b_k, \quad k = 1, 2 && \text{on } \Gamma, \\
n \cdot v &= d && \text{on } \Gamma, \\
\rho &= \rho_{in} && \text{on } \Gamma_{in},
\end{aligned} \tag{1.1}$$

where  $v : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is the unknown velocity field of the fluid and  $\rho : \mathbb{R}^3 \rightarrow \mathbb{R}$  is the unknown density. We assume that the pressure is a function of the density of a class  $C^3$ . Further,  $\mu$  and  $\nu$  are viscosity coefficients satisfying  $(\mu + 2\nu) > 0$  and  $f > 0$  is a friction coefficient. The domain  $\Omega$  is a cylinder in  $\mathbb{R}^3$  of a form  $\Omega = \Omega_0 \times (0, L)$  where  $\Omega_0 \in \mathbb{R}^2$  is a set with a boundary regular enough and  $L$  is a positive constant (see fig. 1). We want to show existence of a solution that can be considered a small perturbation of a constant flow  $(\bar{v}, \bar{\rho}) \equiv ([1, 0, 0], 1)$ . Thus we denote the subsets of the boundary  $\Gamma = \partial\Omega$  as  $\Gamma = \Gamma_{in} \cup \Gamma_{out} \cup \Gamma_0$ , where  $\Gamma_{in} = \{x \in \Gamma : \bar{v} \cdot n < 0\}$ ,  $\Gamma_{out} = \{x \in \Gamma : \bar{v} \cdot n > 0\}$  and  $\Gamma_0 = \{x \in \Gamma : \bar{v} \cdot n = 0\}$ .

By  $n$  we denote the outward unit normal to  $\Gamma$  and  $\tau_1, \tau_2$  are the unit tangent vectors to  $\Gamma$ . Since the boundary has singularities at the junctions of  $\Gamma_{in}$  and  $\Gamma_{out}$  with  $\Gamma_0$ , for the boundary traces we will consider functional spaces that are algebraic sums of spaces defined on the boundary. More precisely for  $s, q \in \mathbb{R}$  we shall denote  $W_s^q(\Gamma) := W_s^q(\Gamma_{in}) + W_s^q(\Gamma_{out}) + W_s^q(\Gamma_0)$ . We

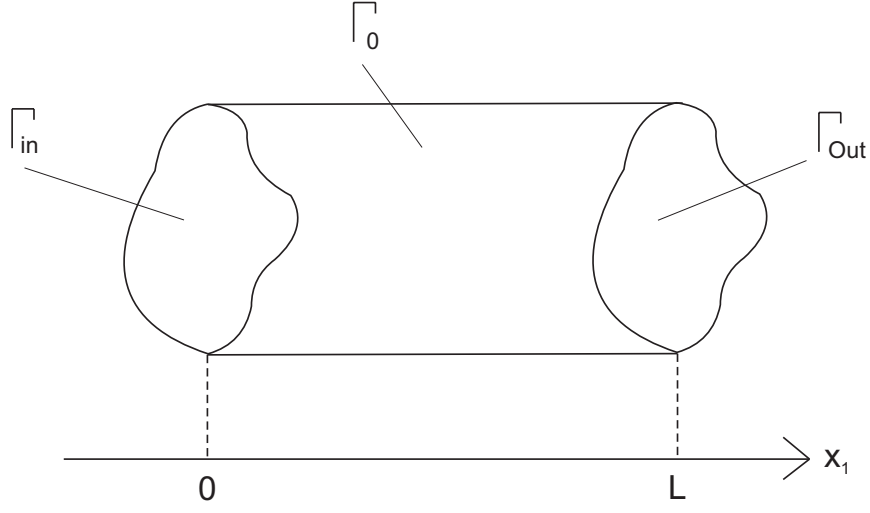


Figure 1: The domain

assume that  $b \in W_p^{1-1/p}(\Gamma)$ ,  $\rho_{in} \in W_p^1(\Gamma_{in})$  and  $d \in W_p^{2-1/p}(\Gamma)$  are given functions and  $d = 0$  on  $\Gamma_0$  what means that  $\Gamma_0$  is an impermeable wall.

For simplicity we consider the momentum equation with zero r.h.s., but our proofs work without any modification for the r.h.s.  $\rho F$  where  $F$  is small enough in  $L_p$ .

We shall make here some remarks concerning notation. Since we will usually use the spaces of functions defined on  $\Omega$ , we will skip  $\Omega$  in notation of the spaces, for example we will write  $L_2$  instead of  $L_2(\Omega)$ . For the density we will use estimates in the space  $L_\infty(0, L; L_2(\Omega_0))$ . For simplicity we will denote this space by  $L_\infty(L_2)$ . A constant dependent on the data that can be controlled, but not necessarily small, will be denoted by  $C$ , and  $E$  shall denote a constant that can be arbitrarily small provided that the data is small enough.

In order to formulate our main result let us define a quantity  $D_0$  that measures how the boundary data  $b, d$  and  $\rho_{in}$  differ from the values of, respectively,  $f\bar{v} \cdot \tau_i$ ,  $n \cdot \bar{v}$  and  $\bar{\rho}$  in appropriate norms. We have  $\bar{v} \cdot \tau_i = \tau_i^{(1)}$  and  $\bar{v} \cdot n = n^{(1)}$ , thus we define

$$D_0 = \|b_i - f\tau_i^{(1)}\|_{W_p^{1-1/p}(\Gamma)} + \|d - n^{(1)}\|_{W_p^{2-1/p}(\Gamma)} + \|\rho_{in} - 1\|_{W_p^1(\Gamma_{in})}. \quad (1.2)$$

Our main result is

**Theorem 1.** *Assume that  $D_0$  defined in (1.2) is small enough,  $f$  is large enough and  $p > 3$ . Then there exists a solution  $(v, \rho) \in W_p^2(\Omega) \times W_p^1(\Omega)$  to the system (1.1) and*

$$\|v - \bar{v}\|_{W_p^2} + \|\rho - \bar{\rho}\|_{W_p^1} \leq E(D_0), \quad (1.3)$$

where  $E(D_0)$  can be arbitrarily small provided that  $D_0$  is small enough. This solution is unique in the class of solutions satisfying the estimate (1.3).

The major difficulty in the proof of Theorem 1 is in the term  $u \cdot \nabla w$  in the continuity equation, that yields impossible a direct application of a fixed point argument. To overcome this

problem one can apply the method of elliptic regularization, known rather from the theory of weak solutions (see [16]). This method has been applied to a similar problem in a two dimensional case in [18]. However, it complicates considerably the computations since we have to find the bound on the artificial diffusive term. Here we apply a method of successive approximations, that leads to a more direct proof. In order to prove Theorem 1 we will construct a sequence  $(u^n, w^n) \in W_p^2 \times W_p^1$  that converges to the solution of (1.1). Due to the presence of the term  $u \cdot \nabla w$  we can not show directly the convergence in  $W_p^2 \times W_p^1$ , but we can show that  $(u^n, w^n)$  is a Cauchy sequence in a larger space  $H^1 \times L_\infty(L_2)$  and thus converges in this space to the weak solution of (1.1). On the other hand, the sequence will converge on a subsequence weakly in  $W_p^2 \times W_p^1$ , what will enable us to show that the weak solution is in fact strong. A similar approach has been applied in [4] to an evolutionary Navier-Stokes system in a framework of Besov spaces.

We start with removing the inhomogeneity from the boundary condition (1.1)<sub>4</sub>. To this end let us construct  $u_0 \in W_p^2(\Omega)$  such that

$$n \cdot u_0|_\Gamma = d - n^{(1)}. \quad (1.4)$$

Due to the assumption of smallness of  $d - n^{(1)}$  in  $W_p^{2-1/p}(\Gamma)$  we can assume that

$$\|u_0\|_{W_p^2} < 1. \quad (1.5)$$

From now on we assume (1.5) in all our results. Now we consider

$$u = v - \bar{v} - u_0 \quad \text{and} \quad w = \rho - \bar{\rho}.$$

One can easily verify that  $(u, w)$  satisfies the following system:

$$\begin{aligned} \partial_{x_1} u - \mu \Delta u - (\nu + \mu) \nabla \operatorname{div} u + \pi'(1) \nabla w &= F(u, w) & \text{in } \Omega, \\ \operatorname{div} u + \partial_{x_1} w + (u + u_0) \cdot \nabla w &= G(u, w) & \text{in } \Omega, \\ n \cdot 2\mu \mathbf{D}(u) \cdot \tau_i + f u \cdot \tau_i &= B_i, \quad i = 1, 2 & \text{on } \Gamma, \\ n \cdot u &= 0 & \text{on } \Gamma, \\ w &= w_{in} & \text{on } \Gamma_{in}, \end{aligned} \quad (1.6)$$

where

$$\begin{aligned} F(u, w) &= -w(u + \bar{v} + u_0) \cdot \nabla(u + u_0) - (u_0 \cdot \nabla u) - u \cdot \nabla u_0 \\ &\quad + \mu \Delta u_0 + (\nu + \mu) \nabla \operatorname{div} u_0 - u_0 \cdot \nabla u_0 - [\pi'(w + 1) - \pi'(1)] \nabla w, \\ G(u, w) &= -(w + 1) \operatorname{div} u_0 - w \operatorname{div} u \end{aligned} \quad (1.7)$$

and

$$B_i = b_i - 2\mu n \cdot \mathbf{D}(u_0) \cdot \tau_i - f \tau_i^{(1)}.$$

From now on we will denote  $\pi'(1) =: \gamma$ . We see that  $F$  and  $G$  also depend on  $\nabla u, u_0, \nabla u_0$ , but for simplicity we will write  $F(u, w)$  and  $G(u, w)$ . In order to prove Theorem 1 it is enough to show the existence of a solution  $(u, w)$  to the system (1.6) provided that  $\|B\|_{W_p^{1-1/p}(\Gamma)}$  and

$\|u_0\|_{W_p^2(\Omega)}$  are small enough. As we already mentioned, we will construct a sequence that converges to the solution. The sequence will be defined as

$$\begin{aligned} \partial_{x_1} u^{n+1} - \mu \Delta u^{n+1} - (\nu + \mu) \nabla \operatorname{div} u^{n+1} + \gamma \nabla w^{n+1} &= F(u^n, w^n) && \text{in } \Omega, \\ \operatorname{div} u^{n+1} + \partial_{x_1} w^{n+1} + (u^n + u_0) \cdot \nabla w^{n+1} &= G(u^n, w^n) && \text{in } \Omega, \\ n \cdot 2\mu \mathbf{D}(u^{n+1}) \cdot \tau_i + f u^{n+1} \cdot \tau_i &= B_i, \quad i = 1, 2 && \text{on } \Gamma, \\ n \cdot u^{n+1} &= 0 && \text{on } \Gamma, \\ w^{n+1} &= w_{in} && \text{on } \Gamma_{in}. \end{aligned} \quad (1.8)$$

As we will see in the sequel, our method does not require any particular starting point for the sequence  $(u^n, w^n)$ , but only some smallness assumptions on the starting point  $(u^0, w^0)$ , hence without loss of generality we can set  $(u^0, w^0) = (0, 0)$ . In order to show the existence of the sequence defined in (1.8) we have to solve a linear system:

$$\begin{aligned} \partial_{x_1} u - \mu \Delta u - (\nu + \mu) \nabla \operatorname{div} u + \gamma \nabla w &= F && \text{in } \Omega, \\ \operatorname{div} u + \partial_{x_1} w + (\bar{u} + u_0) \cdot \nabla w &= G && \text{in } \Omega, \\ n \cdot 2\mu \mathbf{D}(u) \cdot \tau_i + f u \cdot \tau_i &= B_i, \quad i = 1, 2 && \text{on } \Gamma, \\ n \cdot u &= 0 && \text{on } \Gamma, \\ w &= w_{in} && \text{on } \Gamma_{in}, \end{aligned} \quad (1.9)$$

where  $(F, G, \bar{u}, u_0) \in L_p \times W_p^1 \times W_p^2 \times W_p^2$  are given functions and  $\bar{u} \cdot n = 0$  on  $\Gamma$ .

Let us now outline the strategy of the proof, and thus the structure of the paper. In section 2 we show the *a priori* estimate (2.35) on a solution to the linear system (1.9). We start with an energy estimate in  $H^1 \times L_\infty(L_2)$ . Next the properties of the slip boundary conditions enables us to show that the vorticity of the velocity on the boundary has the same regularity as the velocity, and this fact makes it possible to find a bound on  $\|w\|_{W_p^1}$ . Then the estimate (2.35) results directly from the elliptic regularity of the Lamé system.

The linear system (1.9) is solved in section 3. First we show the existence of a weak solution using the Galerkin method modified to deal with the continuity equation. Next we can show that this solution is in fact strong using *a priori* estimate and symmetry of the slip boundary conditions.

In section 4 we show the estimate in  $W_p^2 \times W_p^1$  on the sequence  $(u^n, w^n)$  and, as a result, the Cauchy condition satisfied by this sequence in the space  $H^1 \times L_\infty(L_2)$ . These results are derived by application of the estimates for the linear system.

In section 5 we apply the results of section 4 passing to the limit with  $(u^n, w^n)$  and then showing that the limit is a solution to (1.6). Finally we show that this solution is unique in a class of solutions satisfying the estimate (1.3).

## 2 A priori bounds

The main result of this section is the estimate (2.35) in  $W_p^2 \times W_p^1$ . In order to show it we start with an energy estimate in  $H^1 \times L_\infty(L_2)$ . Next we consider the equation on the vorticity of the velocity and apply the Helmholtz decomposition to derive the bound on  $\|w\|_{W_p^1}$  and finally using the classical elliptic theory we conclude (2.35).

In our proofs we shall not need explicit formulas on the functions  $F(u, w)$  and  $G(u, w)$ , what will be important is that they depend quadratically on  $u$  and  $w$ . More precisely, we will show a following estimate

**Lemma 1.** *Let  $(u, w) \in W_p^2 \times W_p^1$  and let  $F(u, w)$  and  $G(u, w)$  be defined in (1.7). Then*

$$\begin{aligned} & \|F(u, w)\|_{L_p} + \|G(u, w)\|_{W_p^1} \leq \\ & \leq C [(\|u\|_{W_p^2} + \|w\|_{W_p^1})^2 + \|u_0\|_{W_p^2}^2]. \end{aligned} \quad (2.1)$$

**Proof** Since by the imbedding theorem  $W_p^1(\Omega) \subset L_\infty(\Omega)$ , the estimate on  $\|G\|_{W_p^1}$  is straightforward, and the only part of  $F$  that deserves attention is  $\delta\pi'(w)\nabla w$ , where

$$\delta\pi'(w) := \pi'(w+1) - \pi'(1). \quad (2.2)$$

We will apply a fact that for a  $C^1$  - function  $f$  we have

$$f(x) - f(y) = (x - y) \int_0^1 f'[tx + (1-t)y] dt, \quad (2.3)$$

Thus we have

$$\delta\pi'(w) = w \int_0^1 \pi''(tw + 1) dt.$$

Since  $\pi$  is a  $C^3$  - function, the above implies

$$\|\delta\pi'(w)\nabla w\|_{L_p} \leq C(\pi)\|w\|_\infty\|\nabla w\|_{L_p} \leq C\|w\|_{W_p^1}^2.$$

The other parts of  $F$  can be estimated directly giving (2.1).  $\square$

Next, we derive the 'energy' estimate in  $H^1 \times L_\infty(L_2)$ . It is stated in the following lemma

**Lemma 2.** *Let  $(u, w)$  be a solution to the system (1.9) with  $(F, G, B, w_{in}, \bar{u}) \in V^* \times L_2 \times L_2(\Gamma) \times L_2(\Gamma_{in}) \times W_p^2$ , with  $\|\bar{u}\|_{W_p^2}$  small enough and  $f$  large enough. Then*

$$\|u\|_{H^1} + \|w\|_{L_\infty(L_2)} \leq C [\|F\|_{V^*} + \|G\|_{L_2} + \|B\|_{L_2(\Gamma)} + \|w_{in}\|_{L_2(\Gamma_{in})}], \quad (2.4)$$

where

$$V = \{v \in H^1(\Omega) : v \cdot n|_\Gamma = 0\} \quad (2.5)$$

and  $V^*$  is the dual space of  $V$ .

**Proof.** We apply a general identity

$$\begin{aligned} & \int_\Omega (-\mu\Delta u - (\nu + \mu)\nabla \operatorname{div} u) v dx = \\ & = \int_\Omega \{2\mu\mathbf{D}(u) : \nabla v + \nu \operatorname{div} u \operatorname{div} v\} dx - \int_\Gamma n \cdot [2\mu\mathbf{D}(u) + \nu \operatorname{div} u \mathbf{Id}] \cdot v d\sigma. \end{aligned} \quad (2.6)$$

For  $u, v$  satisfying the boundary conditions (1.9)<sub>3,4</sub> the boundary term in (2.6) equals

$$\int_\Gamma \left\{ \sum_{i=1}^2 [B_i - f(u \cdot \tau_i)] (v \cdot \tau_i) \right\} d\sigma.$$

Thus multiplying (1.9)<sub>1</sub> by  $u$  and integrating over  $\Omega$  we get

$$\begin{aligned} \int_{\Omega} \{2\mu \mathbf{D}^2(u) + \nu \operatorname{div}^2 u\} dx + \int_{\Gamma} (f + \frac{n^{(1)}}{2}) |u|^2 d\sigma - \gamma \int_{\Omega} w \operatorname{div} u dx = \\ = \int_{\Omega} F \cdot u dx + \int_{\Gamma} \{B_1(u \cdot \tau_1) + B_2(u \cdot \tau_2)\} d\sigma. \end{aligned} \quad (2.7)$$

From now on (not only in this proof but also later) we will use the summation convention when taking the sum over the tangential components. Applying (1.9)<sub>2</sub> and the boundary conditions we get

$$\begin{aligned} \int_{\Omega} w \operatorname{div} u dx &= \int_{\Omega} G w dx + \frac{1}{2} \int_{\Omega} w^2 \operatorname{div} (\bar{u} + u_0) dx \\ &- \frac{1}{2} \int_{\Gamma_{out}} w^2 (1 + u_0^{(1)}) d\sigma + \frac{1}{2} \int_{\Gamma_{in}} w_{in}^2 (1 + u_0^{(1)}) d\sigma. \end{aligned}$$

For  $\|u_0\|_{W_p^2}$  small enough we have by the imbedding theorem  $1 + u_0^{(1)} > 0$  a.e. on  $\Gamma_{out}$  what yields  $\int_{\Gamma_{out}} w^2 (1 + u_0^{(1)}) d\sigma > 0$ . Moreover, for the friction  $f$  large enough on  $\Gamma_{in}$  the boundary term in (2.7) will be positive. Combining these facts with the Korn inequality (that can be proved in a simple way with the friction large enough - see Lemma 2.4 in [17]):

$$\int_{\Omega} 2\mu \mathbf{D}^2(u) + \int_{\Gamma} f(u \cdot \tau)^2 d\sigma \geq C \|u\|_{H^1}^2 \quad (2.8)$$

we derive from (2.7) the following inequality

$$C \|u\|_{H^1}^2 \leq \int_{\Omega} F \cdot u dx + \int_{\Gamma} B_i(u \cdot \tau_i) d\sigma + \frac{1}{2} \int_{\Omega} w^2 \operatorname{div} (\bar{u} + u_0) dx - \frac{1}{2} \int_{\Gamma_{in}} w_{in}^2 (1 + u_0^{(1)}) d\sigma. \quad (2.9)$$

In order to derive (2.4) from (2.9) we have to estimate  $\|w\|_{L_{\infty}(L_2)}$  in terms of  $\|u\|_{H^1}$  and the data. To show this estimate we refer to section 3 where the linear system (1.9) is solved. Namely, we have  $w = S(G - \operatorname{div} u)$  where the operator  $S$  is defined in (3.7) and thus the estimate (3.8) implies

$$\|w\|_{L_{\infty}(L_2)} \leq C (\|G\|_{L_2} + \|u\|_{H^1} + \|w_{in}\|_{L_2(\Gamma_{in})}). \quad (2.10)$$

The above inequality combined with (2.9) yields (2.4).  $\square$

Now we consider the vorticity of the velocity  $\alpha = \operatorname{rot} u$ . The properties of the slip boundary conditions enables us to express the tangential components of  $\alpha$  on the boundary in terms of the velocity. We arrive at the following system

$$\begin{aligned} \partial_{x_1} \alpha - \mu \Delta \alpha &= \operatorname{rot} F && \text{in } \Omega, \\ \alpha \cdot \tau_2 &= (2\chi_1 - \frac{f}{\nu}) u \cdot \tau_1 + \frac{B_1}{\nu} && \text{on } \Gamma, \\ \alpha \cdot \tau_1 &= (\frac{f}{\nu} - 2\chi_2) u \cdot \tau_2 - \frac{B_2}{\nu} && \text{on } \Gamma, \\ \operatorname{div} \alpha &= 0 && \text{on } \Gamma, \end{aligned} \quad (2.11)$$

where  $\chi_i$  denote the curvatures of the curves generated by tangent vectors  $\tau_i$ . In order to show the boundary relations (2.11)<sub>2,3</sub> it is enough to differentiate (1.9)<sub>4</sub> with respect to the tangential directions and apply (1.9)<sub>3</sub>. A rigorous proof, modifying the proof in the two-dimensional case

from [15], is given in the Appendix. The condition  $\operatorname{div} \alpha = 0$  in  $\Omega$  results simply from the fact that  $\alpha = \operatorname{rot} u$ . We introduce this relation as a boundary condition  $(2.11)_4$ , that completes the conditions on the tangential parts of the vorticity. What is remarkable in the boundary conditions  $(2.11)_{2,3}$  is that the tangential parts of the vorticity on the boundary has the same regularity as the velocity itself and the data. This feature of slip boundary conditions makes it possible to show the higher estimate on the vorticity (see [12],[13], [23]).

In order to derive the bound on the vorticity we can follow [23], Lemma 4, and construct  $\alpha_0$ , a divergence-free extension of the boundary data  $(2.11)_{2,3}$ , for example as a solution to the Stokes problem with zero r.h.s and the boundary conditions  $(2.11)_{2,3}$  supplied with  $\alpha_0 \cdot n = 0$ . The theory of the Stokes system then yields

$$\|\alpha_0\|_{W_p^1} \leq C [\|u\|_{W_p^{1-1/p}(\Gamma)} + \|B\|_{W_p^{1-1/p}(\Gamma)}]. \quad (2.12)$$

Then the function  $\alpha - \alpha_0$  satisfies the system

$$\begin{aligned} -\mu \Delta (\alpha - \alpha_0) &= \operatorname{rot} [F - \partial_{x_1} u] + \mu \Delta \alpha_0 && \text{in } \Omega, \\ (\alpha - \alpha_0) \cdot \tau_1 &= 0 && \text{on } \Gamma, \\ (\alpha - \alpha_0) \cdot \tau_2 &= 0 && \text{on } \Gamma, \\ \operatorname{div} (\alpha - \alpha_0) &= 0 && \text{on } \Gamma. \end{aligned} \quad (2.13)$$

Here we have used the fact that  $\partial_{x_1} \alpha = \operatorname{rot} \partial_{x_1} u$  to preserve the rotational structure of the r.h.s. For the above system we have the following estimate (see [26])

$$\|\alpha\|_{W_p^1} \leq C [\|F\|_{L_p} + \|\partial_{x_1} u\|_{L_p} + \|\alpha_0\|_{W_p^1}]. \quad (2.14)$$

The term with  $\alpha_0$  can be bounded by (2.12) and to deal with  $\partial_{x_1} u$  we apply the interpolation inequality (6.3). We obtain the term  $\|u\|_{H^1}$  that we bound using (2.4) and finally arrive at

$$\|\alpha\|_{W_p^1} \leq C(\epsilon) [\|F\|_{L_p} + \|G\|_{W_p^1} + \|w_{in}\|_{L_2(\Gamma_{in})} + \|u\|_{W_p^{1-1/p}(\Gamma)} + \|B\|_{W_p^{1-1/p}(\Gamma)}] + \epsilon \|u\|_{W_p^2}. \quad (2.15)$$

With the bound on the vorticity at hand the next step is to consider the Helmholtz decomposition of the velocity (the proof can be found in [7]):

$$u = \nabla \phi + A, \quad (2.16)$$

where  $\phi|_{\Gamma} = 0$  and  $\operatorname{div} A = 0$ . We see that the field  $A$  satisfies the following system

$$\begin{aligned} \operatorname{rot} A &= \alpha && \text{in } \Omega, \\ \operatorname{div} A &= 0 && \text{in } \Omega, \\ A \cdot n &= 0 && \text{on } \Gamma. \end{aligned} \quad (2.17)$$

This is the standard rot-div system and we have  $\|A\|_{W_p^2} \leq C \|\alpha\|_{W_p^1}$ , what by (2.15) can be rewritten as

$$\|A\|_{W_p^2} \leq C(\epsilon) [\|F\|_{L_p} + \|G\|_{W_p^1} + \|u\|_{W_p^{1-1/p}(\Gamma)} + \|B\|_{W_p^{1-1/p}(\Gamma)} + \|w_{in}\|_{W_p^1(\Gamma_{in})}] + \epsilon \|u\|_{W_p^2} \quad (2.18)$$



for any  $\epsilon > 0$ . Now we substitute the Helmholtz decomposition to (1.9)<sub>1</sub>. We get

$$\nabla[-(\nu + 2\mu)\Delta\phi + \gamma w] = F - \partial_{x_1}A + \mu\Delta A + (\nu + \mu)\nabla \operatorname{div} A - \partial_{x_1}\phi, \quad (2.19)$$

but  $\Delta\phi = \operatorname{div} u$  and denoting the l.h.s. of the above equation by  $\bar{F}$  we obtain

$$-(\nu + 2\mu)\operatorname{div} u + \gamma w = \bar{H}, \quad (2.20)$$

where  $\nabla \bar{H} = \bar{F}$ . Combining the last equation with (1.9)<sub>2</sub> we arrive at

$$\bar{\gamma}w + w_{x_1} + (\bar{u} + u_0)\nabla w = H, \quad (2.21)$$

where  $\bar{\gamma} = \frac{\gamma}{\nu+2\mu}$  and

$$H = \frac{\bar{H}}{\nu + 2\mu} + G. \quad (2.22)$$

The equation (2.21) makes it possible to estimate the  $W_p^1$ -norm of the density in terms of  $W_p^1$ -norm of  $H$ . The latter will be controlled since (2.19) enables us to bound  $\|\nabla H\|_{L_p}$  and  $\|H\|_{L_p}$  using interpolation and the energy estimate (2.4). The details are presented in the proof of lemma 4, but first we estimate  $\|w\|_{W_p^1}$  in terms of  $H$ . The result is stated in the following lemma

**Lemma 3.** *Assume that  $w$  satisfies the equation (2.21) with  $H \in W_p^1$ . Then*

$$\|w\|_{W_p^1} \leq C [\|H\|_{W_p^1} + \|w_{in}\|_{W_p^1(\Gamma_{in})}]. \quad (2.23)$$

**Proof.** In order to find a bound on  $\|w\|_{L_p}$  we multiply (2.21) by  $|w|^{p-2}w$  and integrate over  $\Omega$ . Integrating by parts and next using the boundary conditions we get

$$\int_{\Omega} |w|^{p-2}w w_{x_1} dx = \frac{1}{p} \int_{\Omega} \partial_{x_1} |w|^p dx = \frac{1}{p} \int_{\Gamma_{out}} |w|^p d\sigma - \frac{1}{p} \int_{\Gamma_{in}} |w|^p d\sigma,$$

since  $n^{(1)} \equiv 0$  on  $\Gamma_0$ ,  $n^{(1)} \equiv -1$  on  $\Gamma_{in}$  and  $n^{(1)} \equiv 1$  on  $\Gamma_{out}$ . Similarly, applying the boundary conditions we get

$$\begin{aligned} \int_{\Omega} (\bar{u} + u_0) \cdot (|w|^{p-2}w \nabla w) dx &= \frac{1}{p} \int_{\Omega} (\bar{u} + u_0) \cdot \nabla |w|^p dx = \\ &= -\frac{1}{p} \int_{\Omega} \operatorname{div} (\bar{u} + u_0) |w|^p dx + \frac{1}{p} \int_{\Gamma_{out}} u_0^{(1)} |w|^p d\sigma - \frac{1}{p} \int_{\Gamma_{in}} u_0^{(1)} |w|^p d\sigma. \end{aligned}$$

Thus multiplying (2.21) by  $|w|^{p-2}w$  we get

$$\begin{aligned} \bar{\gamma} \|w\|_{L_p}^p - \frac{1}{p} \int_{\Omega} \operatorname{div} (\bar{u} + u_0) |w|^p dx + \frac{1}{p} \int_{\Gamma_{out}} |w|^p (1 + u_0^{(1)}) d\sigma &\leq \\ &\leq \|H\|_{L_p} \|w\|_{L_p}^{p-1} + \frac{1}{p} \int_{\Gamma_{in}} |w_{in}|^p (1 + u_0^{(1)}) d\sigma. \end{aligned} \quad (2.24)$$

By the imbedding theorem the smallness of  $\|\bar{u} + u_0\|_{W_p^2}$  implies  $1 + u_0^{(1)} > 0$  a.e. in  $\Omega$  and  $\bar{\gamma} - \|\operatorname{div} (u + u_0)\|_{\infty} > 0$ . Thus the boundary term on the l.h.s. is positive and the term with  $\operatorname{div} (u + u_0)$  can be combined with the first term of the l.h.s, what yields

$$C \|w\|_{L_p}^p \leq \|H\|_{L_p} \|w\|_{L_p}^{p-1} + C \|w_{in}\|_{L_p(\Gamma_{in})}^p,$$

and so

$$\|w\|_{L_p} \leq C [\|H\|_{L_p} + \|w_{in}\|_{L_p(\Gamma_{in})}]. \quad (2.25)$$

The derivatives of the density are estimated in a similar way. In order to find a bound on  $w_{x_i}$  we differentiate (2.21) with respect to  $x_i$ . If we assume that  $w \in W_p^1$  then (2.21) implies  $\tilde{u} \cdot \nabla w \in W_p^1$ , where

$$\tilde{u} := [1 + (\bar{u} + u_0)^{(1)}, (\bar{u} + u_0)^{(2)}, (\bar{u} + u_0)^{(3)}]. \quad (2.26)$$

Thus  $\tilde{u} \cdot \nabla w_{x_i} := (\tilde{u} \cdot \nabla w)_{x_i} - \tilde{u}_{x_i} \cdot \nabla w \in L_p$ . Hence we can differentiate (2.21) with respect to  $x_i$ , multiply by  $|w_{x_i}|^{p-2} w_{x_i}$  and integrate. Since  $\tilde{u}_{x_i} = (\bar{u} + u_0)_{x_i}$ , we have

$$\int_{\Omega} \tilde{u}_{x_i} \cdot (|w_{x_i}|^{p-2} w_{x_i} \nabla w) dx \leq \|\nabla(\bar{u} + u_0)\|_{L_{\infty}} \|\nabla w\|_{L_p}^p \leq C \|\bar{u} + u_0\|_{W_p^2} \|\nabla w\|_{L_p}.$$

Next, since  $\tilde{u} \cdot \nabla w_{x_i} \in L_p$ , we can write

$$\begin{aligned} \int_{\Omega} \tilde{u} \cdot |w_{x_i}|^{p-2} w_{x_i} \nabla w_{x_i} dx &= \frac{1}{p} \int_{\Omega} \tilde{u} \cdot \nabla |w_{x_i}|^p dx = -\frac{1}{p} \int_{\Omega} |w_{x_i}|^p \operatorname{div} \tilde{u} dx + \frac{1}{p} \int_{\Gamma} |w_{x_i}|^p \tilde{u} \cdot n d\sigma = \\ &= -\frac{1}{p} \int_{\Omega} |w_{x_i}|^p \operatorname{div} \tilde{u} dx - \frac{1}{p} \int_{\Gamma_{in}} |w_{in,x_i}|^p (1 + u_0^{(1)}) d\sigma + \frac{1}{p} \int_{\Gamma_{out}} |w_{x_i}|^p (1 + u_0^{(1)}) d\sigma. \end{aligned}$$

For  $i = 2, 3$  we have  $w_{in,x_i} \in L_p(\Gamma_{in})$  and hence the above defines the trace of  $|w_{x_i}|^p$  on  $\Gamma_{out}$ . We arrive at

$$\begin{aligned} \bar{\gamma} \|w_{x_i}\|_{L_p}^p - \frac{1}{p} \int_{\Omega} \operatorname{div}(\bar{u} + u_0) |w_{x_i}|^p dx + \frac{1}{p} \int_{\Gamma_{out}} |w_{x_i}|^p (1 + u_0^{(1)}) d\sigma &\leq \\ \leq \|H_{x_i}\|_{L_p} \|w_{x_i}\|_{L_p}^{p-1} + \frac{1}{p} \int_{\Gamma_{in}} |w_{in,x_i}|^p (1 + u_0^{(1)}) d\sigma + C \|\bar{u} + u_0\|_{W_p^2} \|\nabla w\|_{L_p}^p. \end{aligned} \quad (2.27)$$

For  $i = 2, 3$  it gives directly the bound on  $\|w_{x_i}\|_{L_p}$ . In order to estimate  $w_{x_1}$  we start the same way differentiating (2.21) with respect to  $x_1$  and multiplying by  $|w_{x_1}|^{p-2} w_{x_1}$ . The difference in comparison to  $w_{x_2}$  and  $w_{x_3}$  is that  $w_{x_1}$  is not given on  $\Gamma_{in}$ . In order to overcome this difficulty we can observe that on  $\Gamma_{in}$  the equation (2.21) reduces to

$$\bar{\gamma} w_{in} + (\bar{u} + u_0)^{(2)} w_{in,x_2} + (\bar{u} + u_0)^{(3)} w_{in,x_3} + [1 + (\bar{u} + u_0)^{(1)}] w_{x_1} = H,$$

what can be rewritten as

$$w_{x_1} = \frac{1}{1 + (\bar{u} + u_0)^{(1)}} [H - \bar{\gamma} w_{in} - (\bar{u} + u_0)_{\tau} \cdot \nabla_{\tau} w_{in}].$$

Thus we have

$$\|w_{x_1}\|_{L_p(\Gamma_{in})} \leq C [\|H|_{\Gamma_{in}}\|_{L_p(\Gamma_{in})} + \|w_{in}\|_{W_p^1(\Gamma_{in})}].$$

Using this bound in (2.27),  $i = 1$ , we arrive at the estimate

$$\|w_{x_1}\|_{L_p}^p \leq C [\|H_{x_1}\|_{L_p} \|w_{x_1}\|_{L_p}^{p-1} + \|\bar{u} + u_0\|_{W_p^2} \|\nabla w\|_{L_p}^p + \|H\|_{L_p(\Gamma_{in})}^p + \|w_{in}\|_{W_p^1(\Gamma_{in})}^p]. \quad (2.28)$$

The boundary term  $\|H\|_{L_p(\Gamma_{in})}$  can be replaced by  $\|H\|_{W_p^1}$  due to the trace theorem. Thus combining (2.27) (for  $x_2$  and  $x_3$ ) with (2.28) we get

$$\|\nabla w\|_{L_p}^p \leq C [\|\nabla H\|_{L_p} \|\nabla w\|_{L_p}^{p-1} + \|\bar{u} + u_0\|_{W_p^2} \|\nabla w\|_{L_p}^p + \|H\|_{W_p^1}^p + \|w_{in}\|_{W_p^1(\Gamma_{in})}^p]. \quad (2.29)$$

The term  $\|u + u_0\|_{W_p^2} \|\nabla w\|_{L_p}^p$  can be put on the l.h.s. due to the smallness assumption and thus we get

$$\|\nabla w\|_{L_p} \leq C [\|H\|_{W_p^1} + \|w_{in}\|_{W_p^1(\Gamma_{in})}], \quad (2.30)$$

what combined with (2.25) yields

$$\|w\|_{W_p^1} \leq C [\|H\|_{W_p^1} + \|H\|_{L_p(\Gamma_{in})} + \|w_{in}\|_{W_p^1(\Gamma_{in})}]. \quad (2.31)$$

Applying again the trace theorem to the term  $\|H\|_{L_p(\Gamma_{in})}$  we arrive at (2.23).  $\square$

The next step is to estimate  $H$  in terms of the data. The result is in the following

**Lemma 4.** *Let  $H$  be defined in (2.22). Then  $\forall \delta > 0$  we have*

$$\|H\|_{W_p^1} \leq \delta \|u\|_{W_p^2} + C(\delta) [\|F\|_{L_p} + \|G\|_{W_p^1} + \|B\|_{W_p^{1-1/p}(\Gamma)} + \|w_{in}\|_{W_p^1(\Gamma_{in})}]. \quad (2.32)$$

**Proof.** Applying first the interpolation inequality (6.3) and then the estimate (2.4) we get

$$\|H\|_{L_p} \leq \delta_1 \|\nabla H\|_{L_p} + C(\delta_1) [\|F\|_{L_2} + \|G\|_{L_2} + \|B\|_{L_2(\Gamma)}]. \quad (2.33)$$

Next, by (2.19) we have

$$\|\nabla H\|_{L_p} \leq C [\|F\|_{L_p} + \|G\|_{W_p^1} + \|A\|_{W_p^2} + \|\partial_{x_1}\phi\|_{L_p}],$$

where  $u = \nabla\phi + A$  is the Helmholtz decomposition. Now we use the bound (2.18) on  $\|A\|_{W_p^2}$ . We obtain a term  $\|u\|_{W_p^{1-1/p}(\Gamma)}$ , that we estimate using the trace theorem and the interpolation inequality (6.3). The same inequality is applied to estimate  $\|\partial_{x_1}\phi\|_{L_p}$ . We arrive at

$$\begin{aligned} \|\nabla H\|_{L_p} &\leq C [\|F\|_{L_p} + \|G\|_{W_p^1} + \|B\|_{W_p^{1-1/p}(\Gamma)} + \|w_{in}\|_{W_p^1(\Gamma_{in})}] \\ &\quad + \delta_1 \|u\|_{W_p^2} + C(\delta_1) [\|F\|_{L_2} + \|G\|_{L_2} + \|B\|_{L_2(\Gamma)}]. \end{aligned} \quad (2.34)$$

Combining (2.33) and (2.34) we get (2.32)  $\square$ .

Now we are ready to show the *a priori* estimate in  $W_p^2 \times W_p^1$  on the solution of the linear problem.

**Lemma 5.** *Let  $(u, w)$  be a solution to (1.9) with  $(F, G, B, w_{in}, \bar{u}) \in L_p \times W_p^1 \times W_p^{1-1/p}(\Gamma) \times W_p^1(\Gamma_{in}) \times W_p^2$ , with  $\|\bar{u}\|_{W_p^2}$  small enough and  $f$  large enough. Then*

$$\|u\|_{W_p^2} + \|w\|_{W_p^1} \leq C [\|F\|_{L_p} + \|G\|_{W_p^1} + \|B\|_{W_p^{1-1/p}(\Gamma)} + \|w_{in}\|_{W_p^1(\Gamma_{in})}]. \quad (2.35)$$

**Proof.** If  $(u, w)$  is a solution to (1.9), then in particular the velocity satisfies the Lamé system

$$\begin{aligned} \partial_{x_1} u - \mu \Delta u - (\nu + \mu) \nabla \operatorname{div} u &= F - \gamma \nabla w & \text{in } \Omega, \\ n \cdot 2\mu \mathbf{D}(u) \cdot \tau_i + f u \cdot \tau_i &= B_i, \quad i = 1, 2 & \text{on } \Gamma, \\ n \cdot u &= 0 & \text{on } \Gamma. \end{aligned} \quad (2.36)$$

The classical theory of elliptic equations (Agmon, Douglis, Nirenberg [2], [3]) yields

$$\|u\|_{W_p^2} \leq C [\|F\|_{L_p} + \|w\|_{W_p^1} + \|B\|_{W_p^{1-1/p}} + \|u\|_{W_p^1}].$$

Applying the interpolation inequality (6.3) to the term  $\|u\|_{W_p^1}$  and then the energy estimate (2.4) we get

$$\|u\|_{W_p^2} \leq C [\|F\|_{L_p} + \|G\|_{W_p^1} + \|w\|_{W_p^1} + \|B\|_{W_p^{1-1/p}} + \|w_{in}\|_{L_2(\Gamma_{in})}]. \quad (2.37)$$

In order to complete the proof we combine (2.23) and (2.32) obtaining

$$\|w\|_{W_p^1} \leq \delta \|u\|_{W_p^2} + C(\delta) [\|F\|_{L_p} + \|G\|_{W_p^1} + \|B\|_{W_p^{1-1/p}(\Gamma)} + \|w_{in}\|_{W_p^1(\Gamma_{in})}], \quad (2.38)$$

and choosing for example  $\delta = \frac{1}{2C}$  where  $C$  is the constant from (2.37) we arrive at (2.35).  $\square$

### 3 Solution of the linear system

In this section we show the existence of the sequence  $(u^n, w^n)$  defined in (1.8). To this end we have to solve the linear system (1.9) where  $(F, G, \bar{u}, u_0) \in L_p \times W_p^1 \times W_p^2 \times W_p^2$  are given functions such that  $\bar{u} \cdot n = 0$  on  $\Gamma$ . First we apply the Galerkin method to prove the existence of a weak solution and next we show that this solution is strong. For simplicity we will denote  $\bar{u} + u_0$  by  $\bar{u}$ .

#### 3.1 Weak solution

Let us recall the definition of the space  $V$  (2.5). A natural definition of a weak solution to the system (1.9) is a couple  $(u, w) \in V \times L_\infty(L_2)$  such that

$$\begin{aligned} \int_{\Omega} \{v \cdot \partial_{x_1} u + 2\mu \mathbf{D}(u) : \nabla v + \nu \operatorname{div} u \operatorname{div} v - \gamma w \operatorname{div} v\} dx + \int_{\Gamma} f(u \cdot \tau_i)(v \cdot \tau_i) d\sigma = \\ = \int_{\Omega} F \cdot v dx + \int_{\Gamma} B_i(v \cdot \tau_i) d\sigma \end{aligned} \quad (3.1)$$

is satisfied  $\forall v \in V$  and  $(1.9)_2$  is satisfied in  $\mathcal{D}'(\Omega)$ , i.e.  $\forall \phi \in \bar{C}^\infty(\Omega)$ :

$$- \int_{\Omega} w \tilde{u} \cdot \nabla \phi dx - \int_{\Omega} w \phi \operatorname{div} \tilde{u} dx + \int_{\Gamma_{out}} w \phi d\sigma = \int_{\Omega} \phi (G - \operatorname{div} u) dx + \int_{\Gamma_{in}} w_{in} \phi d\sigma, \quad (3.2)$$

where  $\tilde{u}$  is defined in (2.26). Let us introduce an orthonormal basis of  $V$ :  $\{\omega_i\}_{i=1}^\infty$ . We consider finite dimensional spaces:  $V^N = \{\sum_{i=1}^N \alpha_i \omega_i : \alpha_i \in \mathbf{R}\} \subset V$ . The sequence of approximations to the velocity will be searched for in a standard way as  $u^N = \sum_{i=1}^N c_i^N \omega_i$ . Due to the equation (1.9)<sub>2</sub> we have to define the approximations to the density in an appropriate way. Namely, we set  $w^N = S(G^N - \operatorname{div} u^N)$ , where  $S : L_2(\Omega) \rightarrow L_\infty(L_2)$  is defined as

$$w = S(v) \iff \begin{cases} \partial_{x_1} w + \bar{u} \cdot \nabla w = v & \text{in } \mathcal{D}'(\Omega), \\ w = w_{in} & \text{on } \Gamma_{in}. \end{cases} \quad (3.3)$$

We want the image of  $S$  to be in the space  $L_\infty(L_2)$  so that we can apply the theory of transport equation treating  $x_1$  as a 'time' variable to show that  $S$  is well defined. In order to solve the system on the r.h.s. of (3.3) we can search for a change of variables  $x = \psi(z)$  satisfying the identity

$$\partial_{z_1} = \partial_{x_1} + \bar{u} \cdot \nabla_x. \quad (3.4)$$

We construct the mapping  $\psi$  in the following

**Lemma 6.** *Let  $\|\bar{u}\|_{W_p^2}$  be small enough. Then there exists a set  $U \subset \mathbf{R}^3$  and a diffeomorphism  $x = \psi(z)$  defined on  $U$  such that  $\Omega = \psi(U)$  and (3.4) holds. Moreover, if  $z_n \rightarrow z$  and  $\psi(z_n) \rightarrow \Gamma_0$  then  $n^1(z) = 0$ , where  $n$  is the outward normal to  $U$ .*

Before we start with the proof we shall make one remark. The last condition states that the first component of the normal to  $\psi^{-1}(\Gamma_0)$  vanishes, but since  $\psi$  is defined only on  $U$  we formulate this condition using the limits. It means simply that the image  $U = \psi^{-1}(\Omega)$  is also a cylinder with a flat wall. It will be important in the construction of the operator  $S$ .

**Proof of lemma 6.** The identity (3.4) means that  $\psi$  must satisfy

$$\frac{\partial \psi^1}{\partial z_1} = 1 + \bar{u}^1(\psi), \quad \frac{\partial \psi^2}{\partial z_1} = \bar{u}^2(\psi), \quad \frac{\partial \psi^3}{\partial z_1} = \bar{u}^3(\psi). \quad (3.5)$$

A natural condition is that  $\psi(\Gamma_{in}) = \Gamma_{in}$ . Thus we can search for  $\psi(z_1, z_2, z_3) = \psi_{z_2, z_3}(z_1)$ , where for all  $(z_2, z_3)$  such that  $(z_2, z_3, 0) \in \Gamma_{in}$  the function  $\psi_{z_2, z_3}(\cdot)$  is a solution to a system of ODE:

$$\begin{cases} \partial_s \psi_{z_2, z_3}^1 = 1 + \bar{u}^1(\psi_{z_2, z_3}), & \partial_s \psi_{z_2, z_3}^2 = \bar{u}^2(\psi_{z_2, z_3}), & \partial_s \psi_{z_2, z_3}^3 = \bar{u}^3(\psi_{z_2, z_3}), \\ \psi_{z_2, z_3}(0) = (0, z_2, z_3). \end{cases} \quad (3.6)$$

The r.h.s of the system (3.6) is a Lipschitz function with a constant  $K = \|\nabla \bar{u}\|_\infty$  and thus provided that  $\|\bar{u}\|_{W_p^2}$  is small enough the system (3.6) has a unique solution defined on some interval  $(0, b_{z_1, z_2})$ , where  $b_{z_1, z_2}$  depends on  $z_2, z_3$  and  $\|\nabla \bar{u}\|_\infty$ . Provided that the latter is small enough the function  $\psi(z) = \psi_{z_2, z_3}(z_1)$  will be defined on  $U$  such that  $\Omega = \psi(U)$ .

Now we show that  $\psi(z) = \psi_{z_2, z_3}(z_1)$  is a diffeomorphism. The derivatives with respect to  $z_1$  are given by (3.5) and the remaining derivatives can be expressed in terms of  $\bar{u}$  so we can see that  $J\psi = 1 + E(\bar{u})$ , where  $E(\bar{u})$  is small (and thus  $J\psi > 0$ ) provided that  $\|\bar{u}\|_{W_p^2}$  is small.

To see that  $\psi$  is 1-1 we can write it in a form  $\psi(z) = z + \epsilon(z)$ , where  $\|\nabla \epsilon\|_{L_\infty}$  is small. Assume that  $\psi(z^1) = \psi(z^2)$  and  $z^1 \neq z^2$ . Then there exists  $i$  such that  $|z_i^1 - z_i^2| \geq \frac{1}{3}|z^1 - z^2|$  (the

lowercase denotes the coordinate). On the other hand, we have  $|z_i^1 - z_i^2| = |\epsilon_i(z^1) - \epsilon_i(z^2)| \leq \|\nabla \epsilon\|_{L_\infty} |z^1 - z^2|$ , what contradicts the smallness of  $\|\nabla \epsilon\|_{L_\infty}$ .

We have shown that the mapping  $\psi$  given by (3.6) is a diffeomorphism defined on  $U$  such that  $\psi(U) = \Omega$ . Let us denote  $\phi = \psi^{-1}$ . Now it is natural to define the subsets of  $\partial_U$  as  $\partial_U = U_{in} \cup U_{out} \cup U_0$  where  $U_{in} = \Gamma_{in}$ ,  $U_{out} = \{z : z = \lim \phi(x_n), x_n \rightarrow \Gamma_{out}\}$  and  $U_0 = \{z : z = \lim \phi(x_n), x_n \rightarrow \Gamma_0\}$ .

In order to complete the proof we have to show that  $n^1(z) = 0$  for  $z \in U_0$ . But to this end it is enough to observe that

$$D\psi(z)([1, 0, 0]) = [1 + \bar{u}^1(x), \bar{u}^2(x), \bar{u}^3(x)],$$

where  $x = \psi(z)$ . But for  $x \in \Gamma_0$  the vector on the r.h.s is tangent to  $\Gamma_0$  since  $\bar{u} \cdot n = 0$ . We can conclude that on  $U_0$  the image in  $\psi$  of a straight line  $\{(s, z_2, z_3) : s \in (0, b)\}$  is a curve tangent to  $\Gamma_0$ , and thus  $U_0$  is a sum of such lines and so we have  $n^1(z) = 0$ . The proof of lemma 6 is completed.  $\square$ .

Now we can define  $S(v)$  for a continuous function  $v$  as

$$S(v)(x) = w_{in}(0, \phi_2(x), \phi_3(x)) + \int_0^{\phi_1(x)} v(\psi(s, \phi_2(x), \phi_3(x))) ds. \quad (3.7)$$

The condition  $n^1 = 0$  on  $\phi(\Gamma_0)$  guarantees that a straight line  $(s, z_1, z_2) : s \in (0, b)$  has a picture in  $\Omega$  and thus we integrate along a curve contained in  $\Omega$ . It means that  $S$  is well defined for continuous functions defined on  $\Omega$  and the construction of  $\psi$  clearly ensures that  $S$  satisfies (3.3). Next we have to extend  $S$  on  $L_2(\Omega)$ . To this end we need an estimate in  $L_\infty(L_2)$ . It is given by the following

**Lemma 7.** *Let  $S$  be defined in (3.7). Then*

$$\|S(v)\|_{L_\infty(L_2)} \leq C [\|w_{in}\|_{L_2(\Gamma_{in})} + \|v\|_{L_2(\Omega)}]. \quad (3.8)$$

**Proof.** Let  $\Omega_{x_1}$  denote an  $x_1$  - cut of  $\Omega$  and let  $\bar{x} := (x_2, x_3)$ . Then by (3.7) we have

$$\begin{aligned} \|S(v)\|_{L_2(\Omega_{x_1})}^2 &= \int_{\Omega_{x_1}} \left[ w_{in}(0, \phi_2(x), \phi_3(x)) + \int_0^{\phi_1(x)} v(\psi(s, \phi_2(x), \phi_3(x))) ds \right]^2 d\bar{x} \\ &\leq 2\|w_{in}\|_{L_2(\Gamma_{in})}^2 + C \int_{\Omega_{x_1}} \int_0^{\phi_1(x)} v^2(\psi(s, \phi_2(x), \phi_3(x))) ds d\bar{x} \leq C [\|w_{in}\|_{L_2(\Gamma_{in})}^2 + \|v\|_{L_2(\Omega)}^2]. \end{aligned}$$

The above holds for every  $x_1 \in (0, L)$  what implies (3.8).  $\square$

Now we can define  $S(v)$  for  $v \in L_2(\Omega)$  using a standard density argument. Let us take a sequence of smooth functions  $v_n \rightarrow v$  in  $L_2(\Omega)$ . By (3.8) the sequence  $S(v_n)$  satisfies

$$\|S(v_n)\|_{L_\infty(L_2)} \leq C [\|w_{in}\|_{L_2(\Gamma_{in})} + \sup_n \|v_n\|_{L_2}]. \quad (3.9)$$

The bound on the r.h.s. is uniform in  $n$  and thus  $S(v_n) \rightharpoonup^* \eta$  in  $L_\infty(L_2)$ , and  $\eta$  satisfies the estimate (3.8). In particular for  $\phi \in \bar{C}^\infty(\Omega)$  we have

$$\int_{\Omega} S(v_n) \tilde{u} \cdot \nabla \phi \, dx \rightarrow \int_{\Omega} \eta \tilde{u} \cdot \nabla \phi \, dx \quad \text{and} \quad \int_{\Omega} S(v_n) \phi \operatorname{div} \tilde{u} \, dx \rightarrow \int_{\Omega} \eta \phi \operatorname{div} \tilde{u} \, dx.$$

In order to show that  $\eta = S(v)$ , i.e.  $\eta$  solves the system on the r.h.s. of (3.3) we have to show that  $\int_{\Gamma_{out}} S(v_n) \phi \, d\sigma \rightarrow \int_{\Gamma_{out}} \eta \phi \, d\sigma$ . To this end notice that the proof of lemma 7 implies in particular that  $\|S(v_n)\|_{L_2(\Gamma_{out})}$  satisfies the estimate (3.9). Thus  $S(v_n) \rightharpoonup \zeta$  in  $L_2(\Gamma_{out})$  for some  $\zeta \in L_2(\Gamma_{out})$ , and in particular  $\int_{\Gamma_{out}} S(v_n) \phi \, d\sigma \rightarrow \int_{\Gamma_{out}} \zeta \phi \, d\sigma$ . We have to verify that  $\eta|_{\Gamma_{out}} = \zeta$ . This would not be obvious if we only had  $S(v_n) \in L_\infty(L_2)$ , but indeed the proof of lemma 7 implies a stronger condition that supremum (not only the essential supremum) of  $\|S(v_n)\|_{L_2(\Omega_{x_1})}$  is bounded, thus we must have  $\zeta = \eta|_{\Gamma_{out}}$ . We have shown that  $\tilde{u} \cdot \nabla \eta = v$  in  $\mathcal{D}'(\Omega)$ , thus indeed  $\eta = S(v)$ .

Having the operator  $S$  well defined we are ready to proceed with the Galerkin method. Taking  $F = F^N$ ,  $u = u^N = \sum_i c_i^N \omega_i$ ,  $v = \omega_k$ ,  $k = 1 \dots N$  and  $w = w^N = S(G^N - \operatorname{div} u^N)$  in (3.1), where  $F^N$  and  $G^N$  are orthogonal projections of  $F$  and  $G$  on  $V^N$ , we arrive at a system of  $N$  equations

$$B^N(u^N, \omega_k) = 0, \quad k = 1 \dots N, \quad (3.10)$$

where  $B^N : V^N \rightarrow V^N$  is defined as

$$\begin{aligned} B^N(\xi^N, v^N) = & \int_{\Omega} \{ \xi^N \partial_{x_1} v^N + 2\mu \mathbf{D}(\xi^N) : \nabla v^N + \operatorname{div} \xi^N \operatorname{div} v^N \} \, dx \\ & - \gamma \int_{\Omega} S(G^N - \operatorname{div} \xi^N) \operatorname{div} v^N \, dx + \int_{\Gamma} [f(\xi^N \cdot \tau_j) - B_i] (v^N \cdot \tau_j) \, d\sigma - \int_{\Omega} F^N \cdot v^N \, dx. \end{aligned} \quad (3.11)$$

Now, if  $u^N$  satisfies (3.10) for  $k = 1 \dots N$  and  $w^N = S(G^N - \operatorname{div} u^N)$ , then a pair  $(u^N, w^N)$  satisfies (3.1) - (3.2) for  $(v, \phi) \in (V^N \times \bar{C}^\infty(\Omega))$ . We will call such a pair an approximate solution to (3.1) - (3.2).

The following lemma gives existence of solution to the system (3.10):

**Lemma 8.** *Let  $F, G \in L^2(\Omega)$ ,  $w_{in} \in L_2(\Gamma_{in})$ ,  $B \in L_2(\Gamma)$  and assume that  $f$  is large enough and  $\|\bar{u}\|_{W_p^2}$  is small enough. Then there exists  $u^N \in V^N$  satisfying (3.10) for  $k = 1 \dots N$ . Moreover,*

$$\|u^N\|_{H^1} \leq C(DATA). \quad (3.12)$$

**Proof.** In order to solve the system (3.10) we will apply a well-known result in finite-dimensional Hilbert spaces, lemma 14 in the Appendix. Thus we define the operator  $P^N : V^N \rightarrow V^N$  as

$$P^N(\xi^N) = \sum_k B^N(\xi^N, \omega_k) \omega_k \quad \text{for} \quad \xi^N \in V^N. \quad (3.13)$$

In order to apply lemma 14 we have to show that  $(P(\xi^N), \xi^N) > 0$  on some sphere in  $V^N$ . Since

$B^N(\cdot, \cdot)$  is linear with respect to the second variable, we clearly have

$$\begin{aligned}
(P(\xi^N), \xi^N) &= B^N(\xi^N, \xi^N) = \underbrace{2\mu \int_{\Omega} D^2(\xi^N) dx + \nu \int_{\Omega} \operatorname{div}^2 \xi^N dx}_{I_1} \\
&+ \underbrace{\int_{\Omega} \xi^N \partial_{x_1} \xi^N dx + \int_{\Gamma} f(\xi^N \cdot \tau_i)^2 d\sigma}_{I_2} - \underbrace{\gamma \int_{\Omega} S(G^N - \operatorname{div} \xi^N) \operatorname{div} \xi^N dx}_{I_3} \\
&\quad - \int_{\Omega} F \cdot \xi^N dx - \int_{\Gamma} B_i(\xi^N \cdot \tau_i) d\sigma.
\end{aligned} \tag{3.14}$$

Using the Korn inequality similarly as in the proof of the energy estimate (2.4) we get

$$I_1 + I_2 \geq C \|\xi^N\|_{H^1}^2 \tag{3.15}$$

for  $f$  large enough. We have to find a bound on  $I_3$ . Denoting  $\eta^N = S(G^N - \operatorname{div} \xi^N)$  we have

$$- \int_{\Omega} \eta^N \operatorname{div} \xi^N dx = \int_{\Omega} \eta^N (\partial_{x_1} \eta^N + \bar{u} \cdot \nabla \eta^N) dx - \int_{\Omega} \eta^N G^N dx. \tag{3.16}$$

Using (3.8) we get

$$- \int_{\Omega} \eta^N G^N dx \geq -\|\eta^N\|_{L^2} \|G^N\|_{L^2} \geq -C \|G^N\|_{L^2} (\|G^N\|_{L^2} + \|\xi^N\|_{H^1} + \|w_{in}\|_{L_2(\Gamma_{in})}). \tag{3.17}$$

With the first integral on the r.h.s of (3.16) we have

$$\begin{aligned}
\int_{\Omega} \eta^N (\partial_{x_1} \eta^N + \bar{u} \cdot \nabla \eta^N) dx &= \int_U \eta^N(z) \partial_{z_1} \eta^N(z) J\psi(z) dz = \\
&= \int_U \eta^N(z) \partial_{z_1} \eta^N(z) dz + \int_U \eta^N(z) \partial_{z_1} \eta^N(z) [J\psi(z) - 1] dz.
\end{aligned} \tag{3.18}$$

The first integral can be rewritten as a boundary integral and since  $n^1(z) = 0$  on  $\phi(\Gamma_0)$ , it reduces to

$$\frac{1}{2} \int_{\partial U} [\eta^N(z)]^2 n^1(z) d\sigma(z) = -\frac{1}{2} \int_{U_{in}} [\eta^N(z)]^2 d\sigma(z) + \frac{1}{2} \int_{U_{out}} [\eta^N(z)]^2 d\sigma(z) \geq - \int_{\Gamma_{in}} w_{in}^2 d\sigma(x).$$

In the last passage we used the fact that  $\phi|_{\Gamma_{in}}$  is the identity and that  $n^1(z) > 0$  on  $U_{out}$ , what is true provided that  $\phi$  does not differ too much from the identity on  $\Gamma_{out}$ , what in turn holds under the smallness assumptions on  $\bar{u}$ .

With the second integral on the r.h.s. of (3.18) we have

$$\begin{aligned}
\int_U \eta^N(z) \partial_{z_1} \eta^N(z) [J\psi(z) - 1] dz &\geq -\sup_U |J\psi - 1| \int_U \eta^N(z) (G^N - \operatorname{div}_x \xi^N)(z) dz \geq \\
&\geq -E \|\eta^N\|_{L_2(U)} [\|G^N\|_{L_2(U)} + \|\operatorname{div}_x \xi^N\|_{L_2(U)}] \geq -E [\|G^N\|_{L_2(\Omega)}^2 + \|\xi^N\|_{H^1(\Omega)}^2 + \|w_{in}\|_{L_2(\Gamma_{in})}^2].
\end{aligned}$$

Combining this estimate with (3.15) we get

$$(P^N(\xi^N), \xi^N) \geq C [\|\xi^N\|_{H^1(\Omega)}^2 - D \|\xi^N\|_{H^1(\Omega)} - D^2], \tag{3.19}$$



where  $D = \|F\|_{L^2(\Omega)} + \|G\|_{L^2(\Omega)} + \|w_{in}\|_{L^2(\Gamma_{in})} + \|B\|_{L^2(\Gamma)}$ . Thus there exists  $\tilde{C} = \tilde{C}(\mu, \Omega, D)$  such that  $(P^N(\xi^N), \xi^N) > 0$  for  $\|\xi\| = \tilde{C}$ , and applying lemma 14 we conclude that  $\exists \xi^* : P^N(\xi^*) = 0$  and  $\|\xi^*\| \leq \tilde{C}$ . Moreover, since  $\{\omega_k\}_{k=1}^N$  is the basis of  $V^N$ , we have  $P^N(\xi^*) = 0 \iff (B^N \xi^*, \omega_k) = 0, \quad k = 1 \dots N$ . Thus  $\xi^*$  is a solution to (3.10).  $\square$ .

Now showing the existence of the weak solution is straightforward. The result is in the following

**Lemma 9.** *Assume that  $F, G \in L_2(\Omega)$ ,  $w_{in} \in L_2(\Gamma_{in})$ ,  $B \in L_2(\Gamma)$ . Assume further that  $f$  is large enough and  $\|\bar{u}\|_{W_p^2}$  is small enough. Then there exists  $(u, w) \in V \times W$  that is a weak solution to the system (1.9). Moreover, the weak solution satisfies the estimate (2.4).*

**Proof.** The estimates (3.8) and (3.12) imply that  $\|u^N\|_{H^1} + \|w^N\|_{L_\infty(L_2)} \leq C(DATA)$ . Thus

$$u^N \rightharpoonup u \text{ in } H^1 \quad \text{and} \quad w^N \rightharpoonup^* w \text{ in } L_\infty(L_2)$$

for some  $(u, w) \in H^1 \times L_\infty(L_2)$ . It is very easy to verify that  $(u, w)$  is a weak solution. First, passing to the limit in (3.1) for  $(u^N, w^N)$  we see that  $u$  satisfies (3.1) with  $w$ . On the other hand, taking the limit in (3.2) we verify that  $w = S(G - \operatorname{div} u)$ . We conclude that  $(u, w)$  satisfies (3.1) - (3.2), thus we have the weak solution. To show the boundary condition on the density we can rewrite the r.h.s of (3.3) as

$$\begin{cases} w_{x_1} + \frac{\bar{u}^{(2)}}{1+\bar{u}^{(1)}} w_{x_2} + \frac{\bar{u}^{(3)}}{1+\bar{u}^{(1)}} w_{x_3} = \frac{v}{1+\bar{u}^{(1)}} & \text{in } \mathcal{D}'(\Omega), \\ w = w_{in} & \text{on } \Gamma_{in}, \end{cases} \quad (3.20)$$

and, treating  $x_1$  as a 'time' variable, adapt Di Perna - Lions theory of transport equation ([5]) that implies the uniqueness of solution to (3.20) in the class  $L_\infty(L_2)$ . The proof is thus complete.  $\square$

## 3.2 Strong solution

Having the weak solution of the linear system (1.9) we can show quite easily that this solution is strong if the data has the appropriate regularity. The following lemma gives existence of a strong solution to (1.9).

**Lemma 10.** *Let  $F \in L_p$ ,  $G \in W_p^1$ ,  $w_{in} \in W_p^1(\Gamma_{in})$ ,  $B \in W_p^{1-1/p}(\Gamma)$  and assume that  $f$  is large enough and  $\|\bar{u}\|_{W_p^2}$  is small enough. Then there exist  $(u, w) \in W_p^2 \times W_p^1$  that is a strong solution to (1.9) and satisfies the estimate (2.35).*

**Proof.** Since (1.9) is a linear system, the *a priori* estimate (2.35) will imply the regularity of the weak solution once we can deal with the singularity of the boundary at the junctions of  $\Gamma_0$  with  $\Gamma_{in}$  and  $\Gamma_{out}$ . This however can be done easily since  $\Omega$  is symmetric w.r.t. the plane  $\{x_1 = 0\}$  and the slip boundary conditions preserve this symmetry. More precisely, for  $\{\tilde{x} = (-x_1, x_2, x_3) : x = (x_1, x_2, x_3) \in \Omega\}$  we can consider a vector field

$$\tilde{u}(\tilde{x}) = [-u^1(x), u^2(x), u^3(x)]. \quad (3.21)$$

Then on  $\Gamma_{in}$  we have  $\tilde{u} \cdot n = u \cdot n$  and  $n \cdot \mathbf{D}(\tilde{u}) \cdot \tau_i + \tilde{u} \cdot \tau_i = n \cdot \mathbf{D}(u) \cdot \tau_i + u \cdot \tau_i$ . Hence we can extend the weak solution on the negative values of  $x_1$  using (3.21) and, applying the estimate (2.35), show that the extended solution is in  $W_p^2 \times W_p^1$ . An identical argument can be applied on  $\Gamma_{out}$  and we conclude that  $(u, w)$  is a strong solution to (1.9).  $\square$

## 4 Bounds on the approximating sequence

In this section we will show the bounds on the sequence  $\{(u^n, w^n)\}$  of solutions to (1.8). The term  $u \cdot \nabla w$  in the continuity equation makes it impossible to show directly the convergence in  $W_p^2 \times W_p^1$  to the strong solution of (1.6). We can show however that the sequence of iterated solutions is bounded in  $W_p^2 \times W_p^1$ , and using this bound we can conclude it is a Cauchy sequence in  $H^1 \times L_\infty(L_2)$ , and thus converges in this space to some couple  $(u, w)$ . On the other hand, the boundedness implies weak convergence in  $W_p^2 \times W_p^1$ , and the limit must be  $(u, w)$ . The following lemma gives the boundedness of  $(u^n, w^n)$  in  $W_p^2 \times W_p^1$ .

**Lemma 11.** *Let  $\{(u^n, w^n)\}$  be a sequence of solutions to (1.8) starting from  $(u^0, w^0) = (0, 0)$ . Then*

$$\|u^n\|_{W_p^2} + \|w^n\|_{W_p^1} \leq M, \quad (4.1)$$

where  $M$  can be arbitrarily small provided that  $\|u_0\|_{W_p^2}$  (extension of the boundary data (1.4), not to be confused with  $u^0$  from  $(u^0, w^0)$ , the starting point of the sequence  $(u^n, w^n)$ ),  $\|B\|_{W_p^{1-1/p}(\Gamma)}$ ,  $\|w_{in}\|_{W_p^1(\Gamma_{in})}$  and  $\|\bar{u}\|_{W_p^2}$  are small enough and  $f$  is large enough.

**Proof.** The estimate (2.35) for the iterated system reads

$$\begin{aligned} & \|u^{n+1}\|_{W_p^2} + \|w^{n+1}\|_{W_p^1} \leq \\ & \leq C [\|F(u^n, w^n)\|_{L_p} + \|G(u^n, w^n)\|_{W_p^1} + \|B\|_{W_p^{1-1/p}(\Gamma)} + \|w_{in}\|_{W_p^1(\Gamma_{in})}]. \end{aligned} \quad (4.2)$$

Denoting  $A_n = \|u^n\|_{W_p^2} + \|w^n\|_{W_p^1}$  and  $b = \|u_0\|_{W_p^2} + \|B\|_{W_p^{1-1/p}(\Gamma)} + \|w_{in}\|_{W_p^1(\Gamma_{in})}$ , from (2.1) and (4.2) we get

$$A_{n+1} \leq A_n^2 + b, \quad (4.3)$$

thus  $A_n$  is bounded by a constant that can be arbitrarily small provided that  $A_0$  and  $b$  are small enough. Indeed let us fix  $0 < \delta < \frac{1}{4}$  and assume that  $b < \delta$ . Then (4.3) entails an implication  $A_n \leq 2b \Rightarrow A_{n+1} \leq 2b$  and we can conclude that

$$\left. \begin{array}{l} \delta < \frac{1}{4} \\ b < \delta \\ A_0 < 2b \end{array} \right\} \Rightarrow A_n < 2\delta \quad \forall n \in \mathbb{N}. \quad (4.4)$$

Hence if we fix  $0 < \epsilon < \frac{1}{4}$  and assume that  $\|u_0\|_{W_p^2} + \|B\|_{W_p^{1-1/p}(\Gamma)} + \|w_{in}\|_{W_p^1(\Gamma_{in})} < \epsilon$  then starting the iteration from  $(u^0, w^0) = (0, 0)$  we have

$$\|u^n\|_{W_p^2} + \|w^n\|_{W_p^1} \leq 2\delta \quad \forall n \in \mathbb{N}. \quad \square \quad (4.5)$$

The next lemma almost completes the proof of the Cauchy condition in  $H^1 \times L_\infty(L_2)$  for the sequence of iterated solutions.

**Lemma 12.** *Let the assumptions of Lemma 11 hold. Then we have*

$$\|u^{n+1} - u^{m+1}\|_{H^1} + \|w^{n+1} - w^{m+1}\|_{L_\infty(L_2)} \leq E(M) (\|u^n - u^m\|_{H^1} + \|w^n - w^m\|_{L_\infty(L_2)}), \quad (4.6)$$

where  $M$  is the constant from (4.1).

**Proof.** Subtracting  $(1.8)_m$  from  $(1.8)_n$  we arrive at

$$\begin{aligned} & \partial_{x_1}(u^{n+1} - u^{m+1}) - \mu \Delta(u^{n+1} - u^{m+1}) - (\nu + \mu) \nabla \operatorname{div}(u^{n+1} - u^{m+1}) \\ & \quad + \gamma \nabla(w^{n+1} - w^{m+1}) = F(u^n, w^n) - F(u^m, w^m), \\ & \operatorname{div}(u^{n+1} - u^{m+1}) + \partial_{x_1}(w^{n+1} - w^{m+1}) + (u^n + u_0) \cdot \nabla(w^{n+1} - w^{m+1}) = \\ & \quad = G(u^n, w^n) - G(u^m, w^m) + (u^n - u^m) \cdot \nabla w^m, \\ & n \cdot 2\mu \mathbf{D}(u^{n+1} - u^{m+1}) \cdot \tau_i + f(u^{n+1} - u^{m+1}) \cdot \tau_i|_{\Gamma} = 0, \\ & \quad n \cdot (u^{n+1} - u^{m+1})|_{\Gamma} = 0, \\ & \quad w^{n+1} - w^{m+1}|_{\Gamma_{in}} = 0. \end{aligned}$$

The estimate (2.4) applied to this system yields

$$\begin{aligned} & \|u^{n+1} - u^{m+1}\|_{H^1} + \|w^{n+1} - w^{m+1}\|_{L_{\infty}(L^2)} \leq \\ & \|F(u^n, w^n) - F(u^m, w^m)\|_{V^*} + \|G(u^n, w^n) - G(u^m, w^m)\|_{L_2} + \|(u^n - u^m) \cdot \nabla w^m\|_{L_2}. \end{aligned}$$

In order to derive (4.6) from the above inequality we have to examine the l.h.s. The part with  $G$  is the most straightforward and we have

$$\|G(u^n, w^n) - G(u^m, w^m)\|_{L_2} \leq E(M) (\|u^n - u^m\|_{H^1} + \|w^n - w^m\|_{L_{\infty}(L_2)}). \quad (4.7)$$

The function  $F$  is more complicated and we have to look at the difference more carefully. A direct calculation yields  $F(u^n, w^n) - F(u^m, w^m) = F_1^{n,m} + F_2^{n,m}$ , where

$$\|F_1^{n,m}\|_{V^*} \leq E(M) (\|u^n - u^m\|_{H^1} + \|w^n - w^m\|_{L_{\infty}(L_2)}) \quad (4.8)$$

and

$$F_2^{n,m} = -[\delta\pi'(w^n) - \delta\pi'(w^m)]\nabla w^n + \delta\pi'(w^m)\nabla(w^n - w^m) =: F_{2,1}^{n,m} + F_{2,2}^{n,m}, \quad (4.9)$$

where  $\delta\pi'(\cdot)$  is defined in (2.2). Since we are interested in the  $V^*$ -norm of  $F_2^{n,m}$ , we have to multiply  $F_{2,1}^{n,m}$  and  $F_{2,2}^{n,m}$  by  $v \in V$  and integrate. With  $F_{2,2}^{n,m}$  we get

$$\begin{aligned} & \int_{\Omega} \delta\pi'(w^m) \nabla(w^n - w^m) \cdot v \, dx = \\ & - \int_{\Omega} \delta\pi'(w^m) (w^n - w^m) \operatorname{div} v \, dx - \int_{\Omega} (w^n - w^m) \nabla [\delta\pi'(w^m)] \cdot v \, dx, \end{aligned}$$

and thus we have to estimate  $\delta\pi'(w^m)$  in terms of  $w^m$ . Using (2.3) we can write

$$\delta\pi'(w^m) = w^m \int_0^1 \pi''[tw^m + 1] \, dt, \quad (4.10)$$

what yields  $\|\delta\pi'(w^m)\|_{L_{\infty}} \leq C(\pi)\|w^m\|_{L_{\infty}}$ . Now we have to estimate  $\|\nabla\delta\pi'(w^m)\|_{L_p}$ . Since  $\pi$  is a  $C^3$ -function (and this is the only point where  $C^3$ -regularity is needed) we can take the gradient of (4.10) and verify that  $\|\nabla\delta\pi'(w^m)\|_{L_p} \leq C(\pi)\|\nabla w^m\|_{L_p}$ . Thus we have

$$\begin{aligned} & \left| \int_{\Omega} \delta\pi'(w^m) (w^n - w^m) \operatorname{div} v \, dx \right| \leq \|\delta\pi'(w^m)\|_{L_{\infty}} \|w^n - w^m\|_{L_2} \|\operatorname{div} v\|_{L_2} \leq \\ & \leq C \|w^m\|_{W_p^1} \|w^n - w^m\|_{L_{\infty}(L_2)} \|v\|_V. \end{aligned} \quad (4.11)$$

Next, since  $p > 3$ , by the Sobolev imbedding theorem we have

$$\begin{aligned} & \left| \int_{\Omega} (w^n - w^m) \nabla [\delta\pi'(w^m)] \cdot v \, dx \right| \leq \\ & \leq \|w^n - w^m\|_{L_2} \|\nabla \delta\pi'(w^m)\|_{L_p} \|v\|_{L_6} \leq C \|w^m\|_{W_p^1} \|w^n - w^m\|_{L_{\infty}(L_2)} \|v\|_V \end{aligned} \quad (4.12)$$

Combining (4.11) and (4.12) we get

$$\|F_{2,2}^{n,m}\|_{V^*} \leq E(M) \|w^n - w^m\|_{L_{\infty}(L_2)}. \quad (4.13)$$

In order to estimate  $F_{2,1}^{n,m}$  we will use again (2.3) to write

$$\delta\pi'(w^n) - \delta\pi'(w^m) = (w^n - w^m) \int_0^1 p''[t w^n + (1-t) w^m + 1] \, dt, \quad (4.14)$$

what yields  $\|\delta\pi'(w^n) - \delta\pi'(w^m)\|_{L_2} \leq C \|w^n - w^m\|_{L_2}$ . With this observation we can estimate

$$\begin{aligned} & \left| \int_{\Omega} [\delta\pi'(w^n) - \delta\pi'(w^m)] \nabla w^n \cdot v \, dx \right| \leq \|\delta\pi'(w^n) - \delta\pi'(w^m)\|_{L_2} \|\nabla w^n\|_{L_p} \|v\|_{L_6} \leq \\ & \leq E(\|w^n\|_{W_p^1}) \|w^n - w^m\|_{L_{\infty}(L_2)} \|v\|_V, \end{aligned}$$

what yields

$$\|F_{2,1}^{n,m}\|_{V^*} \leq E(M) \|w^n - w^m\|_{L_{\infty}(L_2)}. \quad (4.15)$$

Combining the estimates on  $F_1^{n,m}$ ,  $F_{2,1}^{n,m}$  and  $F_{2,2}^{n,m}$  we get

$$\|F(u^n, w^n) - F(u^m, w^m)\|_{V^*} \leq E(M) [\|u^n - u^m\|_{H^1} + \|w^n - w^m\|_{L_{\infty}(L_2)}]. \quad (4.16)$$

The part that remains to estimate is  $(u^n - u^m) \cdot \nabla w^m$ . We shall notice here that this is the term which makes it impossible to show the convergence in  $W_p^2 \times W_p^1$  directly. Namely, if we would like to apply the estimate (2.35) to the system for the difference then we would have to estimate  $\|(u^n - u^m) \cdot \nabla w^m\|_{W_p^1}$  what can not be done as we do not have any knowledge about  $\|w\|_{W_p^2}$ .

Fortunately we only need to estimate the  $L_2$ -norm of this awkward term, what is straightforward. Namely, we have

$$\|(u^n - u^m) \cdot \nabla w^m\|_{L_2} \leq \|u^n - u^m\|_{L_q} \|\nabla w^m\|_{L_p} \leq C \|w^m\|_{W_p^1} \|u^n - u^m\|_{H^1}, \quad (4.17)$$

since  $q = \frac{2p}{p-2} < 6$  for  $p < 3$ . We have thus completed the proof of (4.6).  $\square$

Now, lemma 11 implies that the constant  $E(M) < 1$  provided that the data is small enough and the starting point  $(u^0, w^0) = (0, 0)$ . It completes the proof of the Cauchy condition in  $H^1 \times L_{\infty}(L_2)$  for the sequence  $\{(u^n, w^n)\}$ .

**Remark.** Lemmas 11 and 12 hold for any starting point  $(u^0, w^0)$  small enough in  $W_p^2 \times W_p^1$ , not necessarily  $(0, 0)$ , but we can start the iteration from  $(0, 0)$  without loss of generality.

## 5 Proof of Theorem 1

In this section we prove our main result, Theorem 1. First we show existence of the solution passing to the limit with the sequence  $(u^n, w^n)$  and next we show that this solution is unique in the class of solutions satisfying (1.3).

**Existence of the solution.** Since we have the Cauchy condition on the sequence  $(u^n, w^n)$  only in the space  $H^1(\Omega) \times L_\infty(L_2)$ , first we have to show the convergence in the weak formulation of the problem (1.6), transferring the derivatives of the density on the test function. The sequence  $(u^n, w^n)$  satisfies in particular the following weak formulation of (1.8)

$$\begin{aligned} \int_{\Omega} \{v \cdot \partial_{x_1} u^{n+1} + 2\mu \mathbf{D}(u^{n+1}) : \nabla v + \nu \operatorname{div} u^{n+1} \operatorname{div} v - \gamma w^{n+1} \operatorname{div} v\} dx \\ + \int_{\Gamma} f(u^{n+1} \cdot \tau_i) (v \cdot \tau_i) d\sigma = \int_{\Omega} F(u^n, w^n) \cdot v dx + \int_{\Gamma} B_i(v \cdot \tau_i) d\sigma \end{aligned} \quad (5.1)$$

and

$$\begin{aligned} - \int_{\Omega} w^{n+1} [\tilde{u}^n \cdot \nabla \phi + \operatorname{div} \tilde{u}^n \phi] dx + \int_{\Gamma_{out}} w^{n+1} \phi d\sigma = \\ = \int_{\Omega} \phi (G(u^n, w^n) - \operatorname{div} u^{n+1}) dx + \int_{\Gamma_{in}} w_{in} \phi d\sigma \end{aligned} \quad (5.2)$$

$\forall (v, \phi) \in V \times \bar{C}^\infty(\Omega)$ , where  $\tilde{u}^n = [1 + (u^n + u_0)^{(1)}, (u^n + u_0)^{(2)}, (u^n + u_0)^{(3)}]$ .

Now using the convergence in  $H^1 \times L_\infty(L_2)$  combined with the bound (4.1) in  $W_p^2 \times W_p^1$  we can pass to the limit in (5.1) - (5.2). The convergence in all the terms on the r.h.s. of (5.1) is obvious and the only nontrivial step to show the convergence of  $F(u^n, w^n)$  is to show that

$$\int_{\Omega} \delta\pi'(w^n) \nabla w^n \cdot v dx \rightarrow \int_{\Omega} \delta\pi'(w) \nabla w \cdot v dx.$$

To show the above convergence it is enough to verify that

$$\int_{\Omega} [w^n \delta\pi'(w^n) - w \delta\pi'(w)] \operatorname{div} v dx \rightarrow 0 \quad (5.3)$$

and

$$\int_{\Omega} [w^n \nabla \delta\pi'(w^n) - w \nabla \delta\pi'(w)] \cdot v dx \rightarrow 0. \quad (5.4)$$

Applying again (2.3) we have  $\delta\pi'(w^n) - \delta\pi'(w) = (w^n - w) \int_0^1 \pi''(1 + tw^n + (1-t)w) dt$ , hence

$$\|\delta\pi'(w^n) - \delta\pi'(w)\|_{L_2} \leq C \|w^n - w\|_{L_2}, \quad (5.5)$$

what implies directly (5.3). To show (5.4) we integrate by parts arriving at

$$\begin{aligned} \int_{\Omega} [w^n \nabla \delta\pi'(w^n) - w \nabla \delta\pi'(w)] \cdot v dx = - \int_{\Omega} w^n (\delta\pi'(w^n) - \delta\pi'(w)) \operatorname{div} v dx \\ - \int_{\Omega} (\delta\pi'(w^n) - \delta\pi'(w)) v \cdot \nabla w^n dx + \int_{\Omega} (w^n - w) \nabla \delta\pi'(w) \cdot v dx. \end{aligned}$$

Now all the terms on the l.h.s. converge by (5.5), hence we have shown (5.4).

We conclude that  $(u, w)$  satisfies

$$\begin{aligned} & \int_{\Omega} \{v \cdot \partial_{x_1} u + 2\mu \mathbf{D}(u) : \nabla v + \nu \operatorname{div} u \operatorname{div} v - \gamma w \operatorname{div} v\} dx \\ & + \int_{\Gamma} f(u \cdot \tau_i)(v \cdot \tau_i) d\sigma = \int_{\Omega} F(u, w) \cdot v dx + \int_{\Gamma} B_i(v \cdot \tau_i) d\sigma \end{aligned} \quad (5.6)$$

$\forall v \in V$ . In (5.2) we have to check the convergence in the boundary term. We can use the same argument as in the proof of the existence of solution to the linear system when we have passed to the limit with finite dimensional approximations. Namely, in fact  $w^n$  satisfies the Cauchy condition not only in  $L_{\infty}(L_2)$ . A stronger fact holds that  $w^n$  is a Cauchy sequence in  $L_2(\Omega_{x_1})$  for every  $x_1 \in [0, L]$ , where  $\Omega_{x_1}$  denotes the  $x_1$ -cut of  $\Omega$ . In particular  $w^n \rightarrow \zeta$  in  $L_2(\Gamma_{out})$  for some  $\zeta \in L_2(\Gamma_{out})$  and since  $\sup_{x_1 \in [0, L]} \|w\|_{L_2(\Omega_{x_1})} < \infty$  we conclude that  $\zeta = w|_{\Gamma_{out}}$ . This result combined with the obvious convergence of other terms in (5.2) implies

$$- \int_{\Omega} w[\tilde{u} \cdot \nabla \phi + \operatorname{div} \tilde{u} \phi] dx + \int_{\Gamma_{out}} w \phi d\sigma = \int_{\Omega} \phi(G(u, w) - \operatorname{div} u) dx + \int_{\Gamma_{in}} w_{in} \phi d\sigma \quad (5.7)$$

$\forall \phi \in \bar{C}^{\infty}(\Omega)$ , where  $\tilde{u} = [1 + (u + u_0)^{(1)}, (u + u_0)^{(2)}, (u + u_0)^{(3)}]$ .

Hence we have shown that  $(u, w)$  satisfies (5.6) - (5.7), the weak formulation of (1.6). Now we want to show that the strong formulation also holds.

The bound in  $W_p^2 \times W_p^1$  implies  $(u^{n_k}, w^{n_k}) \rightharpoonup (\bar{u}, \bar{w})$  in  $W_p^2 \times W_p^1$  for some  $(\bar{u}, \bar{w}) \in W_p^2 \times W_p^1$ . On the other hand, we have  $(u^{n_k}, w^{n_k}) \rightarrow (u, w)$  in  $H^1 \times L_{\infty}(L_2)$ , thus we conclude that  $(\bar{u}, \bar{w}) = (u, w)$ .

Hence we can integrate by parts in (5.6) - (5.7) to obtain

$$\begin{aligned} & \int_{\Omega} [F(u, w) - \mu \Delta u - (\mu + \nu) \nabla \operatorname{div} u + \gamma \nabla w] \cdot v dx \\ & = \int_{\Gamma} [B_i(v \cdot \tau_i) - n \cdot [2\mu \mathbf{D}(u) + \nu \operatorname{div} u \mathbf{Id}] \cdot v - f(u \cdot \tau_i)(v \cdot \tau_i)] d\sigma \end{aligned} \quad (5.8)$$

and

$$\int_{\Omega} [w_{x_1} + (u + u_0) \cdot \nabla w] \phi dx = \int_{\Omega} [G(u, w) - \operatorname{div} u] \phi dx. \quad (5.9)$$

From these equations we conclude that (1.6)<sub>1,2</sub> are satisfied a.e. in  $\Omega$  and (1.6)<sub>3</sub> is satisfied a.e. on  $\Gamma$ . It remains to verify that (1.6)<sub>4</sub> is satisfied a.e. on  $\Gamma$  and (1.6)<sub>5</sub> holds a.e. on  $\Gamma_{in}$ . The condition (1.6)<sub>4</sub> results from the convergence  $u^n \rightarrow u$  in  $H^1$ .

Finally,  $w^n \rightharpoonup w$  in  $W_p^1$  implies that  $w^n|_{\Gamma_{in}} \rightharpoonup tr w|_{\Gamma_{in}}$  in  $L_p(\Gamma_{in})$ . On the other hand  $w^n|_{\Gamma_{in}} \rightarrow w_{in}$  in  $W_p^1(\Gamma_{in})$  since it is a constant sequence. We conclude that  $w|_{\Gamma_{in}} = w_{in}$ .

**Uniqueness.** In order to prove the uniqueness of the solution consider  $(v_1, \rho_1)$  and  $(v_2, \rho_2)$  being two solutions to (1.1) satisfying (1.3). We will prove that

$$\|v_1 - v_2\|_{H^1}^2 + \|\rho_1 - \rho_2\|_{L_2}^2 = 0. \quad (5.10)$$

For simplicity let us denote  $u := v_1 - v_2$  and  $w := \rho_1 - \rho_2$ . We will show that

$$\|u\|_{H^1} \leq E \|w\|_{L_2} \quad (5.11)$$

and

$$\|w\|_{L_2} \leq C\|u\|_{H^1}, \quad (5.12)$$

what obviously implies (5.10). Subtracting the equations (1.1) for  $(v_1, \rho_1)$  and  $(v_2, \rho_2)$  we get

$$\begin{aligned} w v_2 \cdot \nabla v_2 + \rho_1 u \cdot \nabla v_2 + \rho_1 v_1 \cdot \nabla u - \mu \Delta u - (\mu + \nu) \nabla \operatorname{div} u + I_\pi \nabla w + w \nabla I_\pi &= 0, \\ \rho_1 \operatorname{div} u + w \operatorname{div} v_2 + u \cdot \nabla \rho_2 + v_1 \cdot \nabla w &= 0, \\ n \cdot 2\mu \mathbf{D}(u) \cdot \tau|_\Gamma &= 0, \\ n \cdot u|_\Gamma &= 0, \\ w|_{\Gamma_{in}} &= 0, \end{aligned} \quad (5.13)$$

where

$$I_\pi = \int_0^1 \pi'((t\rho_1) + (1-t)\rho_2) dt. \quad (5.14)$$

Notice that  $I_\pi \in W_p^1$  since  $\rho_i \in W_p^1$  and  $\pi \in C^3$ . In order to show (5.11) we follow the proof of (2.4) multiplying (5.13)<sub>1</sub> by  $\rho_1 u$  (it will be clarified soon why take the test function  $\rho_1 u$  instead of  $u$ ). Using (2.6) we get

$$\begin{aligned} \int_\Omega (2\mu \mathbf{D}^2(u) + \nu \rho_1 \operatorname{div}^2 u) dx &+ \underbrace{\int_\Omega \left\{ 2\mu [(\rho_1 - 1) \mathbf{D}(u) : \nabla u + \mathbf{D}(u) : (u \otimes \nabla \rho_1)] + \nu (\operatorname{div} u) u \cdot \nabla \rho_1 \right\} dx}_{I_1} \\ &- \underbrace{\int_\Omega \left\{ w u \nabla \rho_1 + \rho_1^2 u^2 \cdot \nabla v_2 + u w \rho_1 v_2 \cdot \nabla v_2 \right\} dx}_{I_2} + \underbrace{\int_\Omega \rho_1^2 (v_1 \cdot \nabla u) \cdot u dx}_{I_3} \\ &+ \underbrace{\int_\Omega \rho_1 w u \cdot \nabla I_\pi dx}_{I_4} - \underbrace{\int_\Omega w u \cdot \nabla (I_\pi \rho_1) dx}_{I_5} - \int_\Omega I_\pi w \rho_1 \operatorname{div} u dx + \int_\Gamma \rho_1 f u^2 d\sigma = 0. \end{aligned}$$

We have  $|I_1| + |I_2| \leq E(\|u\|_{H^1}^2 + \|w\|_{L_2}^2)$  and in order to deal with  $I_3$  let us split it into two parts:

$$2I_3 = \underbrace{\int_\Omega \left\{ (\rho_1^2 v_1^{(1)} - 1) \partial_{x_1} |u|^2 + \rho_1^2 v_1^{(2)} \partial_{x_2} |u|^2 + \rho_1^2 v_1^{(3)} \partial_{x_3} |u|^2 \right\} dx}_{I_3^1} + \underbrace{\int_\Omega \partial_{x_1} |u|^2 dx}_{I_3^2}.$$

We have  $|I_3^1| \leq E\|u\|_{H^1}^2$  and  $I_3^2 = \int_\Gamma |u|^2 n^{(1)} d\sigma = -\int_{\Gamma_{in}} |u|^2 d\sigma + \int_{\Gamma_{out}} |u|^2 d\sigma$ . In order to examine  $I_4$  and  $I_5$  we have to differentiate (5.14) what yields

$$\nabla I_\pi = I_\pi^1 \nabla \rho_1 + I_\pi^2 \nabla \rho_2, \quad (5.15)$$

where

$$I_\pi^1 = \int_0^1 \pi''(t\rho_1 + (1-t)\rho_2) t dt \quad \text{and} \quad I_\pi^2 = \int_0^1 \pi''(t\rho_1 + (1-t)\rho_2) (1-t) dt.$$

We have

$$|\int_{\Omega} \rho_1 I_{\pi}^1 u w \nabla \rho_1 dx| \leq \|\rho_1 I_{\pi}^1\|_{L_{\infty}} \|\nabla \rho_1\|_{L_p} \|u\|_{L_6} \|w\|_{L_2} \leq E (\|u\|_{H^1}^2 + \|w\|_{L_2}^2),$$

and the same for  $\int_{\Omega} \rho_1 I_{\pi}^2 u w \nabla \rho_2 dx$ . Thus the application of (5.15) to  $I_4$  yields  $|I_4| \leq E (\|u\|_{H^1}^2 + \|w\|_{L_2}^2)$ . To estimate  $|I_5|$  it is enough to use (5.15) to compute  $\nabla(I_{\pi} \rho_1)$  and then with the same arguments as in case of  $I_4$  we get  $|I_5| \leq E (\|u\|_{H^1}^2 + \|w\|_{L_2}^2)$ . Summarizing our estimates we can write

$$\begin{aligned} \|u\|_{H^1}^2 + \int_{\Gamma_{in}} (\rho_1 f - \frac{1}{2}) |u|^2 d\sigma + \int_{\Gamma_0} \rho_1 f |u|^2 d\sigma + \int_{\Gamma_{out}} (\rho_1 f + \frac{1}{2}) |u|^2 d\sigma \leq \\ \leq \int_{\Omega} I_{\pi} w \rho_1 \operatorname{div} u dx + E \|w\|_{L_2}^2. \end{aligned} \quad (5.16)$$

The boundary integrals over  $\Gamma_0$  and  $\Gamma_{out}$  will be nonnegative for any  $f \geq 0$  and the integral over  $\Gamma_{in}$  will be nonnegative for  $f$  large enough on  $\Gamma_{in}$ . Now in order to obtain (5.11) from (5.16) we can express  $\rho_1 \operatorname{div} u$  in terms of  $w$  using the equation (5.13)<sub>2</sub> (this is why we have tested (5.13)<sub>1</sub> with  $\rho_1 u$  instead of  $u$ ) and rewrite (5.16) as

$$\|u\|_{H^1}^2 \leq - \underbrace{\int_{\Omega} I_{\pi} w^2 \operatorname{div} v_2 dx}_{I_6} - \underbrace{\int_{\Omega} I_{\pi} w u \cdot \nabla \rho_2 dx}_{I_7} - \underbrace{\int_{\Omega} I_{\pi} v_1 w \cdot \nabla w dx}_{I_8} + E \|w\|_{L_2}^2. \quad (5.17)$$

We verify easily that  $|I_6| + |I_7| \leq E (\|u\|_{H^1}^2 + \|w\|_{L_2}^2)$ . We have to put a little more effort to find a bound on  $I_8$ . Let us integrate by parts:

$$2I_8 = \int_{\Omega} I_{\pi} v_1 \nabla w^2 dx = - \int_{\Omega} w^2 \operatorname{div}(I_{\pi} v_1) dx + \int_{\Gamma} w^2 I_{\pi} v_1 \cdot n d\sigma.$$

The boundary term reduces to  $\int_{\Gamma_{out}} I_{\pi} w^2 v_1^{(1)} d\sigma > 0$  and in order to deal with the first term on the l.h.s. notice that

$$\operatorname{div}(I_{\pi} v_1) = \operatorname{div} v_1 I_{\pi} + I_{\pi}^1 v_1 \cdot \nabla \rho_1 + I_{\pi}^2 v_1 \cdot \nabla \rho_2,$$

hence

$$2I_8 \leq - \underbrace{\int_{\Omega} w^2 \operatorname{div} v_1 I_{\pi} dx}_{I_8^1} - \underbrace{\int_{\Omega} w^2 v_1 \cdot \nabla \rho_1 I_{\pi}^1 dx}_{I_8^2} - \underbrace{\int_{\Omega} w^2 v_1 \cdot \nabla \rho_2 I_{\pi}^2 dx}_{I_8^3}.$$

Obviously we have  $|I_8^1| \leq E \|w\|_{L_2}^2$ . In order to bound  $I_8^2$  we can apply the continuity equation that yields  $v_i \cdot \nabla \rho_i = -\rho_i \operatorname{div} v_i$ , what implies  $|I_8^2| = |\int_{\Omega} w^2 \rho_1 \operatorname{div} v_i I_{\pi}^1 dx| \leq E \|w\|_{L_2}^2$ . In the term  $I_8^3$  we can rewrite the mixed component as  $v_1 \cdot \nabla \rho_2 = u \cdot \nabla \rho_2 + v_2 \cdot \nabla \rho_2$  and conclude that  $|I_8^3| \leq E (\|u\|_{H^1}^2 + \|w\|_{L_2}^2)$ . Combining the above results with (5.16) we get (5.11).

In order to show (5.11) we express the pointwise value of  $w$  using (5.13)<sub>2</sub>:

$$w^2(x_1, x_2) = \int_0^{x_1} w w_s(s, x_2) ds = - \int_0^{x_1} \frac{\rho_1}{v_1^{(1)}} w \operatorname{div} u(s, x_2) ds$$



$$\begin{aligned}
& - \int_0^{x_1} \frac{1}{v_1^{(1)}} \left( w^2 \operatorname{div} v_2 + w u \cdot \nabla \rho_2 \right) (s, x_2) ds - \frac{1}{2} \int_0^{x_1} \frac{1}{v_1^{(1)}} [v_1^{(2)} \partial_{x_2} w^2 + v_1^{(3)} \partial_{x_3} w^2] (s, x_2) ds \\
& =: w_1^2 + w_2^2 + w_3^2.
\end{aligned}$$

We estimate directly the first two components of the l.h.s. obtaining

$$\int_{\Omega} w_1^2 dx \leq \epsilon \|w\|_{L_2}^2 + C(\epsilon) \|u\|_{H_1}^2 \quad \forall \epsilon > 0$$

and  $\int_{\Omega} w_2^2 dx \leq E (\|w\|_{L_2}^2 + \|u\|_{H_1}^2)$ . To complete the proof we have to find a bound on  $w_3^2$ . To this end notice that

$$\int_{\Omega} w_3^2 dx = \frac{1}{2} \int_0^L \int_{P_{x_1}} \frac{1}{v_1^{(1)}} [v_1^{(2)} \partial_{x_2} w^2 + v_1^{(3)} \partial_{x_3} w^2] dx dx_1,$$

where  $P_{x_1} = \Omega_0 \times (0, x_1)$ . Integrating by parts in the inner integral we get

$$\int_{\Omega} w_3^2 dx = \frac{1}{2} \int_0^L \left\{ - \int_{P_{x_1}} w^2 \left[ \partial_{x_2} \frac{v_1^{(2)}}{v_1^{(1)}} + \partial_{x_3} \frac{v_1^{(3)}}{v_1^{(1)}} \right] dx + \int_{\partial P_{x_1}} \frac{w^2}{v_1^{(1)}} [v_1^{(2)} n^{(2)} + v_1^{(3)} n^{(3)}] d\sigma \right\} dx_1.$$

The boundary integral reduces to  $\int_{\Gamma_0 \cap \partial P_{x_1}} w^2 v \cdot n d\sigma = 0$ , what implies  $\int_{\Omega} w_3^2 dx \leq E \|w\|_{L_2}^2$  and (5.12) easily follows completing the proof of the uniqueness, and hence the proof of the Theorem.  $\square$

## 6 Appendix

**Vorticity on the boundary.** In order to show the boundary relation (2.11)<sub>3,4</sub> we have to differentiate (1.9)<sub>4</sub> with respect to tangential directions at a given point  $x_0 \in \Gamma$ . Without loss of generality we can assume that  $n(x_0) = (1, 0, 0)$ ,  $\tau_1(x_0) = (0, 1, 0)$  and  $\tau_2(x_0) = (0, 0, 1)$ . Then we can rewrite (1.9)<sub>3</sub> as (all the quantities are taken at  $x_0$ ):

$$\begin{cases} \mu(u^1_{,2} + u^2_{,1}) + f u^2 = B_1, \\ \mu(u^1_{,3} + u^3_{,1}) + f u^3 = B_2. \end{cases} \quad (6.1)$$

Differentiating (1.9)<sub>4</sub> with respect to the tangential direction  $\tau_1$  we get

$$\left( \frac{d}{d\tau_1} n \right) \cdot u + u^1_{,2} = 0. \quad (6.2)$$

If we denote by  $\chi_1$  the curvature of the curve generated by  $\tau_1$  then we have  $\frac{d}{d\tau_1} n = \chi_1 \tau_1$  and (6.2) can be rewritten as  $\chi_1 (\tau_1 \cdot u) + u^1_{,2} = 0$ . Combining this equation with (6.1)<sub>1</sub> we get

$$u^2_{,1} - u^1_{,2} = \left( 2\chi_1 - \frac{f}{\mu} \right) (u \cdot \tau_1) + \frac{B_1}{\mu},$$

what is exactly (2.11)<sub>3</sub>. (2.11)<sub>4</sub> can be shown in the same way differentiating (1.9)<sub>4</sub> with respect to the tangential direction  $\tau_2$ .

**Lemma 13.** (*interpolation inequality*):

$\forall \epsilon > 0 \quad \exists C(\epsilon, p, \Omega)$  such that  $\forall f \in W_p^1(Q)$ :

$$\|f\|_{L_p} \leq \epsilon \|\nabla f\|_{L_p} + C \|f\|_{L_2}. \quad (6.3)$$

**Proof.** Inequality (6.3) results from the inequality  $\|f\|_{L_p} \leq C(p, \Omega) \|f\|_{W_2^1}^\theta \|f\|_{L_2}^{1-\theta}$  for  $2 \leq p < \infty$ , where  $\theta = \frac{n(p-2)}{2p}$  (see [1], Theorem 5.8). Using Cauchy inequality with  $\epsilon$  we get 6.3.  $\square$

The last auxiliary result we use is a following fact on finitely dimensional Hilbert spaces (the proof can be found in [24]):

**Lemma 14.** *Let  $X$  be a finite dimensional Hilbert space and let  $P : X \rightarrow X$  be a continuous operator satisfying*

$$\exists M > 0 : \quad (P(\xi), \xi) > 0 \quad \text{for} \quad \|\xi\| = M. \quad (6.4)$$

*Then  $\exists \xi^* : \quad \|\xi^*\| \leq M \quad \text{and} \quad P(\xi^*) = 0$ .*

**Acknowledgements.** The author would like to thank Piotr Mucha from the University of Warsaw for valuable comments and remarks concerning the paper. The author has been supported by the Polish Ministry of Science and Higher Education grant No. N N201 364736.

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